# Notes on Vector Calculus

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# Chapter 1

# Subsets of Euclidean space, vector fields, and continuity

## Introduction

The aims of this course are the following:

(i) Extend the main results of one-variable Calculus to higher dimensions

(ii) Explore new phenomena which are non-existent in the one-dimensional case

Regarding the first aim, a basic step will be to define notions of continuity and differentiability in higher dimensions. These are not as intuitive as in the one-dimensional case. For example, given a function $f : \mathbb{R} \to \mathbb{R}$, we can say that $f$ is continuous if one can draw its graph $\Gamma_f$ without lifting the pen (resp. chalk) off the paper (resp. blackboard). For any $n \geq 1$, we can still define the graph of a function (here called a *scalar field*) $f : \mathbb{R}^n \to \mathbb{R}$ to be

$$\Gamma(f) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \,|\, y = f(x)\},$$

where $x$ denotes the vector $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$. Since $\mathbb{R}^n \times \mathbb{R}$ is just $\mathbb{R}^{n+1}$, we can think of $\Gamma(f)$ as a subset of the $(n + 1)$-dimensional space. But the graph will be $n$-dimensional, which is hard (for non-constant $f$)to form a picture of, except possibly for $n = 2$; even then it cannot be drawn on a plane like a blackboard or a sheet of paper. So one needs to define basic notions such as continuity by a more formal method. It will be beneficial to think of a lot of examples in dimension 2, where one has some chance of forming a mental picture.

Integration is also subtle. One is able to integrate *nice* functions $f$ on closed rectangular boxes $R$, which naturally generalize the closed interval $[a, b]$ in $\mathbb{R}$, and when there is spherical symmetry, also over closed balls in $\mathbb{R}^n$. Here $f$ being nice means that $f$ is bounded on $R$ and continuous outside a *negligible set*. But it is problematic to define integrals of even continuous functions over arbitrary subsets $Y$ of $\mathbb{R}^n$, even when they are *bounded*, i.e., can be enclosed in a rectangular box. However, when $Y$ is compact, i.e., closed and bounded, one can integrate continuous functions $f$ over it, at least when $f$ vanishes on the *boundary* of $Y$.

The second aim is more subtle than the first. Already in the plane, one is interested in *line integrals*, i.e., integrals over (nice) curves $C$, of *vector fields*, i.e., vectors of scalar fields, and we will be interested in knowing when the integrals depend only on the *beginning and end points* of the curve. This leads to the notion of *conservative fields*, which is very important also for various other subjects like Physics and Electrical Engineering. Somehow the point here is to not blindly compute such integrals, but to exploit the (beautiful) *geometry* of the situation.

This chapter is concerned with defining the basic structures which will be brought to bear in the succeeding chapters. We start by reviewing the *real numbers*.

## 1.1 Construction and properties of real numbers

This section intends to give some brief background for some of the basic facts about real numbers which we will use in this chapter and in some later ones.

Denote by $\mathbb{Z}$ the set of integers $\{0, \pm 1, \pm 2, \ldots\}$ and by $\mathbb{Q}$ the set of rational numbers $\{\frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0\}$. As usual we identify $\frac{a}{b}$ with $\frac{ma}{mb}$ for any integer $m \neq 0$. To be precise, the rational numbers are equivalence classes of ordered pairs $(a, b)$ of integers, $b \neq 0$, with $(a, b)$ being considered equivalent to $(c, d)$ iff $ad = bc$. The rational numbers can be added, subtracted and multiplied, just like the integers, but in addition we can also divide any rational number $x$ by another (non-zero) $y$. The properties make $\mathbb{Q}$ into a *field* (as one says in *Algebra*), not to be confused (at all) with a *vector field* which we will introduce below. One defines the absolute value of any rational number $x$ as $\operatorname{sgn}(x)x$, where $\operatorname{sgn}(x)$ denotes the sign of $x$. Then it is easy to see that $|xy| = |x||y|$ and $|x+y| \leq |x|+|y|$ (triangle inequality). There are other absolute values one can define on $\mathbb{Q}$, and they satisfy a stronger inequality than the triangle inequality. For this reason, some call the one above the *archimedean* absolute value.

The real numbers are not so easy to comprehend, not withstanding the fact that we have

been using them at will and with ease, and their construction is quite subtle. It was a clever ploy on the part of mathematicians to call them real numbers as it makes people feel that they are *real* and so easy to understand. Some irrational numbers do come up in geometry, like the ubiquitous $\pi$, which is the area of a unit circle, and the quadratic irrational $\sqrt{2}$, which is the length of the hypotenuse of a right triangle with two unit sides. However, many irrational real numbers have no meaning except as limits of nice sequences of rational numbers or as points on a continuum.

What constitutes a nice sequence? The basic criterion has to be that the sequence looks like it has a chance of converging, though we should be able to see that without knowing anything about the limit. The precise condition below was introduced by the nineteenth century French mathematician Cauchy, whence the term *Cauchy sequence*.

**Definition**. *A sequence $\{x_1, x_2, \ldots, x_n, \ldots\}$ of rational numbers is Cauchy iff we can find, for every $\epsilon > 0$ in $\mathbb{Q}$, a positive integer $N$ such that $|x_n - x_m| < \epsilon$, for all $n, m \geq N$.*

Simply put, a sequence is Cauchy iff the terms eventually bunch up around each other. This behavior is clearly necessary to have a limit.

Possibly the simplest (non-constant) Cauchy sequence of rational numbers is $\{\frac{1}{n}|n \geq 1\}$. This sequence has a limit in $\mathbb{Q}$, namely 0. (Check it!) An example of a sequence which is not Cauchy is given by $\{x_n\}$, with $x_n = \sum_{k=1}^{n} \frac{1}{k}$. It is not hard to see that this sequence diverges; some would say that the limit is $+\infty$, which lies outside $\mathbb{Q}$.

There are many sequences of rational numbers which are Cauchy but do *not* have a limit in $\mathbb{Q}$. Two such examples $\{x_n|n \geq 1\}$ are given by the following: (i) $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$, and (ii) $x_n = \sum_{k=1}^{n} \frac{1}{k!}$. The first sequence converges to $\pi^2/6$ and the second to $e$, both of which are not rational (first proved by a mathematician named Lambert around 1766). In fact, these numbers are even *transcendental*, which means they are not roots of polynomials $f(X) = a_0 + a_1X + \cdots + a_nX^n$ with $a_0, a_1, \ldots, a_n \in \mathbb{Q}$. Numbers like $\sqrt{2}$ and $i = \sqrt{-1}$ are *algebraic numbers*, though irrational, and are not transcendental. Of course $i$ does not a "real number".

Take your favorite decimal expansion such as the one of $\pi = 3.1415926.....$ This can be viewed as a Cauchy sequence $\{3, 31/10, 314/100, 3141/1000, ...\}$; you may check that such a sequence is Cauchy no matter what the digits are! Again, for "most" expansions you can come up with, the limit is not in $\mathbb{Q}$. Recall that a decimal expansion represents (converges to) a rational number if and only if it is periodic after some point.

Essentially, the construction of $\mathbb{R}$ consists of formally adjoining to $\mathbb{Q}$ the missing limits of Cauchy sequences of rational numbers. How can we do this in a logically satisfactory manner? Since every Cauchy sequence should give a real number and every real number should arise

from a Cauchy sequence, we start with the set $X$ of all Cauchy sequences of rational numbers. On this set we can even define addition and multiplication by $\{x_1, x_2, ..\} + \{y_1, y_2, ...\} := \{x_1 + y_1, x_2 + y_2, ..\}$ etc. and also division if the denominator sequence stays away from zero (there is something to be checked here, namely that for example $\{x_n + y_n\}$ is again Cauchy but this is not so hard). It seems that $X$ with these operations is already something like the real numbers but there is a problem: Two sequences might have the same limit, think of the zero sequence $\{0, 0, 0..\}$ and the reciprocal sequence $\{1/n | n \geq 1\}$, and such sequences should give the same real number, which is 0 in this case. So the last step is to introduce on $X$ an *equivalence relation*: Declare $x = \{x_n\}, y = \{y_n\}$ in $X$ to be equivalent if for any $\epsilon > 0$ in $\mathbb{Q}$, there exists $N > 0$ such that $|x_n - y_n| < \epsilon$ for all $n > N$ (Check that this is indeed an equivalence relation). The equivalence classes of sequences in $X$ are declared to constitute the set $\mathbb{R}$ of all "real numbers". The rational numbers naturally form a subset of $\mathbb{R}$ by viewing $q \in \mathbb{Q}$ as the *class* of the *constant sequence* $\{q, q, \ldots, q, \ldots\}$; note that in this class we have many other sequences such as $\{q + 1/n\}$, $\{q + 2^{-n}\}$, $\{q + 1/n!\}$). Besides $\mathbb{Q}$ we obtain a lot of new numbers to play with. The real number represented by the sequence (ii) above, for example, is called $e$. When we say we have an $x \in \mathbb{R}$ we think of *some* Cauchy sequence $\{x_1, x_2, ..\}$ representing $x$, for example its decimal expansion. But note that our definition of $\mathbb{R}$ is in no way tied to the base ten, only to the rational numbers. Now one has to check that the addition, multiplication and division defined above pass to the equivalence classes, i.e. to $\mathbb{R}$. This is a doable exercise. One can also introduce an order on $X$ which passes to equivalences classes: $\{x_n\} < \{y_n\}$ if there is some $N$ so that $x_n < y_n$ for all $n > N$. So the notion of positivity for rational numbers carries over to the reals. One has the triangle inequality

$$|x + y| \leq |x| + |y|, \quad \text{forall} \quad x, y \in \mathbb{R}.$$

where $|x| = x$, resp. $-x$ if $x \geq 0$, resp. $x < 0$.

Here are *the* key facts about the real numbers on which much of Calculus is based.

**Theorem 1** *(a) (Completeness of $\mathbb{R}$) Every Cauchy sequence of* real *numbers has a limit in $\mathbb{R}$.*

*(b) (Density of the rationals) Every real number is the limit of a Cauchy sequence of rational numbers.*

More precisely, part (a) says that given $\{x_n\} \subset \mathbb{R}$ so that $\forall\ \epsilon > 0,\ \exists\ N$ such that $\forall\ n, m > N,\quad |x_n - x_m| < \epsilon$, then there exists $x \in \mathbb{R}$ so that $\forall\ \epsilon > 0,\ \exists\ N$ such that $\forall\ n > N,\quad |x - x_n| < \epsilon$. In other words, if we repeat the completion process by which we obtained $\mathbb{R}$ from $\mathbb{Q}$ (i.e. start with the set of Cauchy sequences of *real* numbers and pass to equivalence classes) we end up with the real numbers again.

Part (b) says that given any real number $x$ and an $\epsilon > 0$, we can find infinitely many rational numbers in the interval $(x - \epsilon, x + \epsilon)$. In particular, we have the following **very useful fact**:

**Given any pair $x, y$ of real numbers, say with $x < y$, we can find a rational number $z$ such that $x < z < y$, regardless of how small $x - y$ is.**

**Proof of Theorem 1**. Part (b) holds by construction, and it suffices to prove part (a). If $\{x_n\} \subset \mathbb{R}$ is a Cauchy sequence we represent each $x_n$ as a Cauchy sequence of *rational* numbers $x_{n,i}$, say. We can find $f(n) \in \mathbb{N}$ so that $|x_{n,i} - x_{n,j}| < 2^{-n}$ for $i, j \geq f(n)$ because $x_{n,i}$ is Cauchy for fixed $n$. Then the sequence $x_{n,f(n)} \in \mathbb{Q}$ is Cauchy. Indeed, given $\epsilon$ make $n, m$ large enough so that $2^{-n} < \epsilon/3$, $2^{-m} < \epsilon/3$ and $|x_n - x_m| < \epsilon/3$. Now unravel the meaning of this last statement. It means that for $k$ large enough we have $|x_{n,k} - x_{m,k}| < \epsilon/3$. Make $k$ also bigger than $f(n)$ and $f(m)$. Then we have

$$|x_{n,f(n)} - x_{m,f(m)}| < |x_{n,f(n)} - x_{n,k}| + |x_{n,k} - x_{m,k}| + |x_{m,k} - x_{m,f(m)}| < 2^{-n} + \epsilon/3 + 2^{-m} < \epsilon$$

for large enough $n, m$. The real number $x$ represented by the Cauchy sequence $x_{n,f(n)}$ is the limit of the sequence $x_n$. To see this, given $\epsilon$ take $n$ large so that $2^{-n} < \epsilon/2$ and $|x_{k,f(k)} - x_{n,f(n)}| < \epsilon/2$ for $k \geq n$. If we also have $k > f(n)$ then

$$|x_{k,f(k)} - x_{n,k}| \leq |x_{k,f(k)} - x_{n,f(n)}| + |x_{n,f(n)} - x_{n,k}| < \epsilon.$$

By definition this means that $|x - x_n| < \epsilon$.

Done.

The completeness of $\mathbb{R}$ has a number of different but equivalent formulations. Here is one of them. Call a subset $A$ of real numbers *bounded from above* if there is some $y \in \mathbb{R}$ such that $x \leq y$ for every $x$ in $A$. Such a $y$ is called an *upper bound* for $A$. We are interested in knowing if there is a least upper bound. For example, when $A$ is the interval $(a, 5)$ and $a < 5$, $A$ has a least upper bound, namely the number 5. On the other hand, if $a \geq 5$, the set $A$ is empty, and it has no least upper bound. The **least upper bound** is also called the **supremum**, when it exists, and denoted *lub* or *sup*. It is easy to see that the least upper bound has to be unique if it exists.

**Theorem 2** *Let $A$ be a* **non-empty** *subset of $\mathbb{R}$ which is bounded from above. Then $A$ has a least upper bound.*

*Proof.* As we have already mentioned, this property is actually equivalent to the completeness of $\mathbb{R}$. Let's first deduce it from completeness. Since $A$ is bounded from above, there exists,

by definition, some upper bound $b_1 \in \mathbb{R}$. Pick any $a_1 \in A$, which exists because $A$ is non-empty. Consider the midpoint $z_1 = (a_1 + b_1)/2$ between $a_1$ and $b_1$. If $z_1$ is an upper bound of $A$, put $b_2 = z_1$ and $a_2 = a_1$. Otherwise, there is an $a_2 \in A$ so that $a_2 > z_1$ and we put $b_2 = b_1$. In both cases we have achieved the inequality $|b_2 - a_2| \leq \frac{1}{2}|b_1 - a_1|$. Next consider the mid-point $z_2$ between $a_2$ and $b_2$ and define $a_3, b_3$ by the same procedure. Continuing thus "ad infinitum" (to coin a favorite phrase of Fermat), we arrive at *two* sequences of real numbers, namely $\{a_n | n \geq 1\}$ and $\{b_n | n \geq 1\}$. If we put $c = |x_1 - y_1|$, it is easy to see that by construction,

$$|a_n - b_n| \leq \frac{c}{2^{n-1}},$$

and that

$$|a_n - a_{n+1}| \leq |a_n - b_n| \leq \frac{c}{2^{n-1}}$$

and that the same holds for $|b_n - b_{n+1}|$. Consequently, both of these sequences are Cauchy and have the same limit $z \in \mathbb{R}$, say. Now we claim that $z$ is the *lub* of $A$. Indeed, since $z$ is the limit of the upper bounds $b_n$, it must also be an upper bound. On the other hand, since it is also the limit of the numbers $a_n$ lying in $A$, any smaller number than $z$ cannot be an upper bound of $A$. Done.

We only sketch of the proof of the converse. Given a Cauchy sequence $\{x_n\}$ in $\mathbb{R}$ consider the set

$$A = \{y \in \mathbb{R} | \text{ the set } \{n | x_n \leq y\} \text{ is finite, possibly empty}\}.$$

Then $A$ is nonempty because there is $N$ so that $|x_n - x_m| < 1$ for $n, m \geq N$. If $y = x_N - 2$ then $\{n | x_n \leq y\} \subseteq \{1, 2, ..., N\}$ is finite, so $y \in A$. One can then show that a least upper bound of $A$, which will in fact be unique, is also a limit of the sequence $\{x_n\}$.

QED

Finally a word about the **geometric representation of real numbers**. Real numbers $x$ can be represented in a one-to-one fashion by the points $P = P(x)$ on a line, called the *real line* such that the following hold: (i) if $x < y$, $P(y)$ is situated strictly to the right of $P(x)$; and (ii) $|x - y|$ is the distance between $P(x)$ and $P(y)$. In particular, for any pair of real numbers $x, y$, the mid-point between $P(x)$ and $P(y)$, which one can find by a ruler and compass, corresponds to a unique real number $z$ such that $|x - y| = 2|x - z| = 2|y - z|$. It is customary to identify the numbers with the corresponding points, and simply write $x$ to denote both. Note that the notions of line and distance here are classical; in modern, set-theory-based mathematics one simply defines a line as some set of points that is in one-to-one correspondence with the real numbers.

## 1.2 The norm in $\mathbb{R}^n$

Consider the $n$-dimensional Euclidean space

$$\mathbb{R}^n = \{x = (x_1, x_2, \cdots, x_n) \mid x_i \in \mathbb{R}, \forall i\},$$

equipped with the *inner product* (also called the *scalar product*)

$$\langle x, y \rangle \;=\; \sum_{j=1}^{n} x_j y_j,$$

for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, and the *norm* (or *length*) given by

$$||x|| \;=\; \langle x, x \rangle^{1/2} \;\geq\; 0.$$

Note that $\langle x, y \rangle$ is linear in each variable, and that $||x|| = 0$ iff $x = 0$.

**Basic Properties:**

**(i)** $||cx|| = |c|\,||x||$, for all $c \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

**(ii)** (triangle inequality) $||x + y|| \leq ||x|| + ||y||$, for all $x, y \in \mathbb{R}^n$.

**(iii)** (Cauchy-Schwarz inequality) $|\langle x, y \rangle| \leq ||x||||y||$, for all $x, y$ in $\mathbb{R}^n$.

*Proof.* Part (i) follows from the definition. We claim that part (ii) follows from part (iii). Indeed, by the bilinearity and symmetry of $\langle ., . \rangle$,

$$\langle x + y, x + y \rangle \;=\; \langle x, x \rangle + \langle y, y \rangle + 2\langle x, y \rangle.$$

By the Cauchy-Schwartz inequality, $|\langle x, y \rangle| \leq ||x||||y||$, whence the claim.

It remains to prove part (iii). Since the assertion is trivial if $x$ or $y$ is zero, we may assume that $x \neq 0$ and $y \neq 0$. If $w = \alpha x + \beta y$, with $\alpha, \beta \in \mathbb{R}$, we have

$$(*) \qquad 0 \;\leq\; \langle w, w \rangle \;=\; \alpha^2 \langle x, x \rangle + 2\alpha\beta \langle x, y \rangle + \beta^2 \langle y, y \rangle.$$

Since this holds for all $\alpha, \beta$, we are free to choose them. Put

$$\alpha = \langle y, y \rangle, \; \beta = -\langle x, y \rangle.$$

Dividing $(*)$ by $\alpha$, we obtain the inequality

$$0 \leq \langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2.$$

Done.

The *scalar product* $\langle ., .\rangle$ and norm $||.||$ can both be defined on $\mathbb{C}^n$ as well. For this we set

$$\langle x, y\rangle = \sum_{j=1}^{n} x_j \overline{y}_j,$$

where $\overline{z}$ denotes, for any $z \in \mathbb{C}$, the *complex conjugate* of $z$. Here $\langle x, y\rangle$ is linear in the first variable, but *conjugate linear* in the second, i.e., $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$, while $\langle x, \alpha y\rangle = \overline{\alpha}\langle x, y\rangle$, for any complex scalar $\alpha$. In French such a product will be said to be *sesquilinear* (meaning "one and a half" linear). In any case, note that $\langle x, x\rangle$ is a non-negative real number, which is positive iff $x \neq 0$. So again it makes good sense to say that the *norm* (or *length*) of any $x \in \mathbb{C}^n$ to be $||x|| = \sqrt{\langle x, x\rangle}$. It is a routine exercise to verify that the *basic properties* (i), (ii), (iii) above continue to hold in this case.

We may define a sequence of vectors $v_1, v_2, \ldots, v_m, \ldots$ in $\mathbb{R}^n$ a Cauchy sequence iff for every $\varepsilon > 0$, we can find an $N > 0$ such that for all $m, r > N$, $||v_m - v_r|| < \varepsilon$. In other words, the vectors in the sequence eventually become bunched up together, as tightly as one requires.

**Theorem 3** $\mathbb{R}^n$ *is complete with respect to the norm* $||\ ||$.

*Idea of Proof.* The inequality

$$|x_i| = \sqrt{x_i^2} \leq \sqrt{x_1^2 + \cdots + x_n^2} = ||x||$$

shows that the components of any $||\ ||$-Cauchy sequence in $\mathbb{R}^n$ are ordinary Cauchy sequences in $\mathbb{R}$. Hence we are reduced to part (a) of Theorem 1.

## 1.3 Basic Open Sets in $\mathbb{R}^n$

There are (at least) two types of basic "open" sets in $\mathbb{R}^n$, one *round*, using open balls, and the other *flat*, using rectangular boxes.

For each $a \in \mathbb{R}^n$ and $r > 0$ set

$$B_a(r) = \{x \in \mathbb{R}^n \mid ||x - a|| < r\},$$

and call it the *open ball* of radius $r$ and center $a$; and

$$\bar{B}_a(r) = \{x \in \mathbb{R}^n \mid ||x - a|| \leq r\},$$

the *closed ball* of radius $r$ and center $a$. One also defines the *sphere*

$$S_a(r) = \{x \in \mathbb{R}^n | \; ||x - a|| = r\}.$$

A *closed rectangular box* in $\mathbb{R}^n$ is of the form

$$[a, b] \; = \; [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \; = \; \{x \in \mathbb{R}^n \mid \; x_i \in [a_i, b_i], \, \forall i\},$$

for some $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ and $b = (b_1, \cdots, b_n) \in \mathbb{R}^n$.

An *open rectangular box* is of the form

$$(a, b) \; = \; (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \; = \{x \in \mathbb{R}^n \mid \forall i \, x_i \in (a_i, b_i)\}.$$

**Definition**. A basic open set in $\mathbb{R}^{2n}$ is (either) an open ball or an open rectangular box.

**You can use either or both**. One gets the same answers for our problems below. In this vein, observe that every open ball contains an open rectangular box, and conversely.

**Remark**. It is important to note that given any pair of basic open sets $V_1, V_2$, we can find a nonempty basic open set $W$ contained in their intersection $V_1 \cap V_2$ if this intersection is non-empty.

## 1.4 Open and closed sets

Given any subset $X$ of $\mathbb{R}^n$, let us denote by $X^c$ the complement $\mathbb{R}^n - X$ in $\mathbb{R}^n$. Clearly, the complement of the empty set $\emptyset$ is all of $\mathbb{R}^n$.

Let $A$ be a subset of $\mathbb{R}^n$ and let $y$ be a point in $\mathbb{R}^n$. Then there are three possibilities for $y$ relative to $A$.

**(IP)** There exists a basic open set $U$ containing $y$ such that $U \subseteq A$.

**(EP)** There exists a basic open set $U$ centered at $y$ which lies completely in the complement of $A$. i.e., in $\mathbb{R}^n - A$.

**(BP)** Every basic open set centered at $y$ meets both $A$ and $A^c$.

In case (IP), $y$ is called an **interior point of** $A$. In case (EP), $y$ is called an **exterior point of** $A$. In case (BP), $y$ is called a **boundary point of** $A$. Note that in case (IP) $y \in A$, in case (EP) $y \notin A$, and in case (BP) $y$ may or may not belong to $A$.)

**Definition.** *A set $A$ in $\mathbb{R}^n$ is open if and only if every point of $A$ is an interior point.*

Explicitly, this says: "Given any $z \in A$, we can find a basic open set $U$ containing $z$ such that $U \subseteq A$."

**Definition.** $A \subseteq \mathbb{R}^n$ is closed if its complement is open.

**Lemma 1** *A subset $A$ of $\mathbb{R}^n$ is closed iff it contains all of its boundary points.*

*Proof.* Let $y$ be a boundary point of $A$. Suppose $y$ is not in $A$. Then it belongs to $A^c$, which is open. So, by the definition of an open set, we can find a basic open set $U$ containing $y$ with $U \subseteq A^c$. Such a $U$ does not meet $A$, contradicting the condition (BP). So $A$ must contain $y$.

Conversely, suppose $A$ contains all of its boundary points, and consider any $z$ in $A^c$. Then $z$ has to be an interior point or a boundary point of $A^c$. But the latter possibility does not arise as then $z$ would also be a boundary point of $A$ and hence belong to $A$ (by hypothesis). So $z$ is an interior point of $A^c$. Consequently, $A^c$ is open, as was to be shown.

**Examples.** (1) Basic open sets are open: Indeed, let $y$ belong to the open ball $B_a(r) = \{x \mid ||x - a|| < r\}$. Then, since $||\, y - a\,|| < r$, the number $r' = \frac{1}{2}(r - ||\, y - a\,||)$ is positive, and the open ball $B_y(r')$ around $y$ is completely contained in $B_{x_0}(r)$. The case of open rectangular boxes is left as an easy exercise.

(2) The empty set $\phi$ and $\mathbb{R}^n$ are both open and closed.

Since they are complements of each other, it suffices to check that they are both open, which is clear from the definition.

(3) Let $\{W_\alpha\}$ be a (possibly infinite and perhaps uncountable) collection of open sets in $\mathbb{R}^n$. Then their union $W = \cup_\alpha W_\alpha$ is also open.

Indeed, let $y \in W$. Then $y \in W_\alpha$ for some index $\alpha$, and since $W_\alpha$ is open, there is an open set $V \subseteq W_\alpha$ containing $y$. Then we are done as $y \in V \subseteq W_\alpha \subseteq W$.

(4) Let $\{W_1, W_2, \cdots, W_n\}$ be a **finite** collection of open sets. Then their intersection $W = \bigcap_{i=1}^{n} W_i$ is open.

**Proof**. Let $y \in W$. Then $y \in W_i$, $\forall_i$. Since each $W_i$ is open, we can find a basic open set $V_i$ such that $y \in V_i \subseteq W_i$. Then, by the remark at the end of the previous section, we can find a basic open set $U$ contained in the intersection of the $V_i$ such that $y \in U$. Done.

**Warning.** The intersection of an infinite collection of open sets need not be open, already for $n = 1$, as shown by the following (counter)example. Put, for each $k \geq 1$, $W_k = \left(-\frac{1}{k}, \frac{1}{k}\right)$. Then $\cap_k W_k = \{0\}$, which is not open.

(5) Any **finite** set of points $A = \{P_1, \ldots, P_r\}$ is closed.

**Proof**. For each $j$, let $U_j$ denote the complement of $P_j$ (in $\mathbb{R}^n$). Given any $z$ in $U_j$, we can easily find a basic open set $V_j$ containing $z$ which avoids $P_j$. So $U_j$ is open, for each $j$. The complement of $A$ is simply $\cap_{j=1}^{r} U_j$, which is then open by (4).

More generally, one can show, by essentially the same argument, that a *finite* union of closed sets is again closed.

It is important to remember that there are many sets $A$ in $\mathbb{R}^n$ which are neither open nor closed. For example, look at the half-closed, half-open interval $[0, 1)$ in $\mathbb{R}$.

## 1.5 Compact subsets of $\mathbb{R}^n$.

It is easy to check that the closed balls $\overline{B}_a(r)$ and the closed rectangular boxes $[a, b]$ are indeed closed. But they are more than that. They are also bounded in the obvious sense. This leads to the notion of "compactness".

**Definition**. An **open covering** of a set $A$ in $\mathbb{R}^n$ is a collection $\mathcal{U} = \{V_\alpha\}$ of open sets in $\mathbb{R}^n$ such that

$$A \subseteq \cup_\alpha V_\alpha.$$

In other words, each $V_\alpha$ in the collection is open and any point of $A$ belongs to some $V_\alpha$. Note that $\mathcal{U}$ may be infinite, possibly uncountable.

Clearly, any subset $A$ of $\mathbb{R}^n$ admits an open covering. Indeed, we can just take $\mathcal{U}$ to be the singleton $\{\mathbb{R}^n\}$.

A *subcovering* of an opencovering $\mathcal{U} = \{V_\alpha\}$ of a set $A$ is a subcollection $\mathcal{U}'$ of $\mathcal{U}$ such that any point of $A$ belongs to some set in $\mathcal{U}'$.

**Definition.** *A set $A$ in $\mathbb{R}^n$ is compact if and only if* **any** *open covering $\mathcal{U} = \{V_\alpha\}$ of $A$ contains a* **finite** *subcovering.*

*Example of a set which is not compact*: Let $A = (0,1)$ be the open interval in $\mathbb{R}$. Look at $\mathcal{U} = \{W_1, W_2, \cdots\}$ where $W_m = (\frac{1}{m}, 1-\frac{1}{m})$, for each $m$. We claim that $\mathcal{U}$ is an open covering of $A$. Indeed, each $W_m$ is clearly open and moreover, given any number $x \in A$, then $x \in W_m$ for some $m$. But *no* finite subcollection can cover $A$, which can be seen as follows. Suppose $A$ is covered by $W_{m_1}, \ldots, W_{m_r}$ for some $r$, with $m_1 < m_2 < \ldots < m_r$. Then $W_{m_j} \subset W_{m_r}$ for each $j$, while the point $1/(m_r+1)$ belongs to $A$, but not to $W_{m_r}$, leading to a contradiction. Hence $A$ is not compact.

*Example of a set which is compact*: Let $A$ be a finite set $\subseteq \mathbb{R}^n$. Then $A$ is compact. Prove it!

**Theorem 4 (Heine–Borel)** *The closed interval $[a,b]$ in $\mathbb{R}$ is compact for any pair of real numbers $a$, $b$ with $a < b$.*

*Proof.* Let $\mathcal{U} = \{V_\alpha\}$ be an open covering of $[a,b]$. Let us call a subset of $[a,b]$ *good* if it can be covered by a finite number of $V_\alpha$. Put

$$J = \{x \in [a,b] | \, [a,x] \quad \text{is} \quad \text{good}\}.$$

Clearly, $a$ belongs to $J$, so $J$ is non-empty, and by Theorem 1 of section 2.1, $J$ has a least upper bound; denote it by $z$. Since $b$ is an upper bound of $J$, $z \leq b$. We claim that $z$ lies in $J$. Indeed, pick any open set $V_\beta$ in $\mathcal{U}$ which contains $z$. (This is possible because $\mathcal{U}$ is an open covering of $[a,b]$ and $z$ lies in this interval.) Then $V_\beta$ will contain points to the left of $z$; call one of them $y$. Then, by the definition of *lub*, $y$ must lie in $J$ and consequently, $[a,y]$ is covered by a finite subcollection $\{V_{\alpha_1}, \ldots, V_{\alpha_r}\}$ of $\mathcal{U}$. Then $[a,z]$ is covered by $\mathcal{U}' = \{V_\beta, V_{\alpha_1}, \ldots, V_{\alpha_r}\}$, which is still finite; so $z \in J$. In fact, $z$ has to be $b$. Otherwise, the open set $V_\beta$ will also contain points to the right of $z$, and if we call one of them $t$, say, $[a,t]$ will be covered by $\mathcal{U}'$, implying that $t$ lies in $J$, contradicting the fact that $z$ is an upper bound of $J$. Thus $b$ lies in $J$, and the theorem is proved.

Call a subset $A$ in $\mathbb{R}^n$ **bounded** if we can enclose it in a closed rectangular box.

**Theorem 5** *Let $A$ be a subset of $\mathbb{R}^n$ which is closed and bounded. Then $A$ is compact.*

**Corollary 1** *Closed balls and spheres in $\mathbb{R}^n$ are compact.*

**Remark**. It can be shown that the converse of this theorem is also true, i.e., any compact set in $\mathbb{R}^n$ is closed and bounded.

**Proof of Theorem 5**. The first step is to show that **any closed rectangular box $R = [a,b]$ in $\mathbb{R}^n$ is compact**: When $n = 1$, this is just the Heine-Borel theorem. So let $n > 1$ and assume by induction that the assertion holds in dimension $< n$. Now we can write

$$R = [a_1, b_1] \times R', \quad \text{with} \quad R' = [a_2, b_2] \times \ldots \times [a_n, b_n].$$

Let $\mathcal{U} = \{W_\alpha\}$ be an open covering of $R$. Then, for each $y = t \times y'$ in $R$ with $t$ in $[a_1, b_1]$ and $y' = (y_2, \ldots, y_n)$ in $R'$, there is a open set $W_\alpha(y)$ in $\mathcal{U}$ containing this point. By the openness of $W_\alpha(y)$, we can then find an interval $(c_1, d_1) \subset \mathbb{R}$ and an open rectangular box $(c', d') \subset \mathbb{R}^{n-1}$ such that $t \times y \in (c_1, d_1) \times (c', d') \subset W_\alpha(y)$. Then the collection of the sets $(c', d')$ for $y' \in R'$ and $t$ fixed covers $R'$. Since $R'$ is compact by the induction hypothesis, we can find a finite set, call it $\mathcal{V}'$, of the $(c', d')$ whose union covers $R'$. Let $I(t)$ denote the intersection of the corresponding finite collection of open intervals $(c_1, d_1)$, which is open (cf. the previous section) and contains $t$. Then the collection $\{I(t)\}$ is an open covering of $[a_1, b_1]$. By Heine-Borel, we can then extract a finite subcovering, say $\mathcal{V}$, of $[a_1, b_1]$. It is now easy to see that $\mathcal{V} \times \mathcal{V}'$ is a finite subcovering of $[a, b]$. For any $(c_1, d_1) \in \mathcal{V}$ and $(c', d') \in \mathcal{V}'$ we have $(c_1, d_1) \times (c', d') \subseteq W_\alpha$ for some $\alpha$ so $R$ is contained in a union of finitely many $W_\alpha$'s.

The next step is to consider any closed, bounded set $A$ in $\mathbb{R}^n$. Pick any open covering $\mathcal{U}$ of $A$. Since $A$ is bounded, we can enclose it completely in a closed rectangular box $[a, b]$. Since $A$ is closed, its complement $A^c$ is open. Thus $\mathcal{U} \cup \{A^c\}$ is an open covering of $[a, b]$. Then, by the first step, a finite subcollection, say $\{V_{\alpha_1}, \ldots, V_{\alpha_r}, A^c\}$ covers $[a, b]$. Then, since $A^c \cap A = \emptyset$, the (finite) collection $\{V_{\alpha_1}, \ldots, V_{\alpha_r}\}$ covers $A$. Done.

## 1.6 Vector fields and continuity

We are interested in functions:

$$f : \mathcal{D} \to \mathbb{R}^m$$

with $\mathcal{D} \subseteq \mathbb{R}^n$, called the **domain** of $f$. The **image** (or **range**) of $f$ is $f(\mathcal{D}) \subset \mathbb{R}^m$. Such an $f$ is called a **vector field**. When $m = 1$, one says **scalar field** instead, for obvious reasons.

For each $j \leq m$, we write $f_j(x)$ for the $j$th coordinate of $f(x)$. Then $f$ is completely determined by the collection of scalar fields $\{f_j | j = 1, \ldots, m\}$, called the **component fields**.

**Definition.** Let $a \in \mathcal{D}$, $b \in \mathbb{R}^m$. Then $b$ is the **limit** of $f(x)$ as $x$ tends to $a$, denoted

$$b = \lim_{x \to a} f(x),$$

if the following holds: *For any* $\epsilon > 0$ *there is a* $\delta > 0$ *so that for all* $x \in \mathcal{D}$ *with* $||x - a|| < \delta$ *we have* $||f(x) - b|| < \epsilon$.

This is just the like the definition of limits for functions on the real line but with the absolute value replaced by the norm. In dimension 1, the existence of a limit is equivalent to having a right limit and a left limit, and then having the two limits being equal. In higher dimensions, one can approach a point $a$ from infinitely many directions, and one way to think of it will be to start with an open neighborhood of $a$ and then shrinking it in many different ways to the point $a$. So the existence of a limit is more stringent a condition here.

The definition of continuity is now literally the same as in the one variable case:

**Definition.** Let $a$ be in the domain $\mathcal{D}$. Then $f(x)$ is **continuous** at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

*Remarks*:

a) A vector field $f$ is continuous at $a$ iff each component field $f_j$ is continuous, for $j = 1, \ldots, m$.

b) In these definitions we need not assume that $a$ is an interior point of $\mathcal{D}$. For example, $a$ could be a boundary point of a domain $\mathcal{D}$ which is a closed box. In an extreme case $a$ could also be the only point of $\mathcal{D}$ in some open ball. In this case continuity becomes an empty condition; every function $f$ is continuous at such a "discrete" point $a$.

**Examples**:

(1) Let $f(x, y, z) = ((x^2 + y^2 x)z^2, xy + yz)$, $\mathcal{D} = \mathbb{R}^3$. Then $f : \mathbb{R}^3 \to \mathbb{R}^2$ is continuous at any $a$ in $\mathbb{R}^n$.

More generally, **any** polynomial function is $n$ variables is continuous everywhere. Rational functions $\dfrac{P(x_1, \cdots, x_n)}{Q(x_1, \cdots, x_n)}$ are continuous at all $x = (x_1, \cdots, x_n)$ where $Q(x) \neq 0$.

(2) $f$ is a polynomial function of sines, cosines and exponentials.

It is reasonable to ask at this point what all this has to do with open sets and compact sets. We answer this in the following two lemmas. We call a subset of $\mathcal{D} \subseteq \mathbb{R}^n$ open, resp. closed, if it is the intersection of an open, resp. closed set of $\mathbb{R}^n$ with $\mathcal{D}$.

**Lemma 2** *Let $f : \mathcal{D} \to \mathbb{R}^m$ be a vector field. Then $f$ is continuous at every point $a \in \mathcal{D}$ if and only if the following holds: For every open set $W$ of $\mathbb{R}^m$, its inverse image $f^{-1}(W) := \{x \in \mathcal{D} | f(x) \in W\} \subseteq \mathcal{D}$ is open.*

*Warning:* $f$ continuous does *not* mean that the image of an open set is open. Take, for instance, the constant function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 0$, for all $x \in \mathbb{R}$. This is a continuous map, but any open interval $(a, b)$ in $\mathbb{R}$ gets squished to a point, and $f((a, b))$ is not open.

*Proof.* Let $W$ be an open set in $\mathbb{R}^m$ and $a \in f^{-1}(W)$. Then choose $\epsilon$ so that $B_{f(a)}(\epsilon) \subseteq W$ which is possible since $W$ is open. By continuity there is a $\delta$ so that $f(B_a(\delta) \cap \mathcal{D}) \subseteq B_{f(a)}(\epsilon) \subseteq W$ which just means that $B_a(\delta) \cap \mathcal{D} \subseteq f^{-1}(W)$. Since $a$ is arbitrary we find that $f^{-1}(W)$ is open. Conversely, if $f$ satisfies this condition then $f^{-1}(B_f(a)(\epsilon))$ is open since $B_f(a)(\epsilon)$ is open. Hence we can find a small ball $B_a(\delta) \cap \mathcal{D} \subseteq f^{-1}(B_f(a)(\epsilon))$ around $a \in f^{-1}(B_f(a)(\epsilon))$ which implies that $f$ is continuous at $a$.

*Remark:* This Lemma shows that the notion of continuity does not depend on the particular norm function used in its definition, only on the collection of open sets defined via this norm function (recall the equivalent ways of using boxes or balls to define open sets).

**Lemma 3** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Then, given any compact set $C$ of $\mathbb{R}^n$, $f(C)$ is compact.*

*Proof.* Let $C$ be a compact subset of $\mathbb{R}^n$. Pick any open covering $\mathcal{U} = \{W_\alpha\}$ of $f(C)$. Then by the previous lemma, if we put $V_\alpha = f^{-1}(W_\alpha)$, each $V_\alpha$ is an open subset of $\mathbb{R}^n$. Then the collection $\{V_\alpha\}$ will be an open covering of $C$. By the compactness of $C$, we ca then find a finite subcovering $\{V_{\alpha_1}, \ldots, V_{\alpha_r}\}$ of $C$. Since each $W_\alpha$ is simply $f(V_\alpha)$, $f(C)$ will be covered by the finite collection $\{W_{\alpha_1}, \ldots, W_{\alpha_r}\}$. Done.

As a consequence, any continuous image of a closed rectangular box or a closed ball or a sphere will be compact.

# Chapter 2

# Differentiation in higher dimensions

## 2.1 The Total Derivative

Recall that if $f : \mathbb{R} \to \mathbb{R}$ is a 1-variable function, and $a \in \mathbb{R}$, we say that $f$ is differentiable at $x = a$ if and only if the ratio $\frac{f(a+h)-f(a)}{h}$ tends to a finite limit, denoted $f'(a)$, as $h$ tends to 0.

There are two possible ways to generalize this for vector fields

$$f : \mathcal{D} \to \mathbb{R}^m,\ \mathcal{D} \subseteq \mathbb{R}^n,$$

for points $a$ in the *interior* $\mathcal{D}^0$ of $\mathcal{D}$. (The interior of a set $X$ is defined to be the subset $X^0$ obtained by removing all the boundary points. Since every point of $X^0$ is an interior point, it is open.) The reader seeing this material for the first time will be well advised to stick to vector fields $f$ with domain all of $\mathbb{R}^n$ in the beginning. Even in the one dimensional case, if a function is defined on a closed interval $[a, b]$, say, then one can properly speak of differentiability only at points in the open interval $(a, b)$.

The first thing one might do is to fix a vector $v$ in $\mathbb{R}^n$ and saythat $f$ is **differentiable along** $v$ iff the following limit makes sense:

$$\lim_{h\to 0} \frac{1}{h} \left( f(a + hv) - f(a) \right).$$

When it does, we write $f'(a; v)$ for the limit. Note that this definition makes sense because $a$ is an interior point. Indeed, under this hypothesis, $\mathcal{D}$ contains a basic open set $U$ containing $a$, and so $a + hv$ will, for small enough $h$, fall into $U$, allowing us to speak of $f(a + hv)$. This

derivative behaves exactly like the one variable derivative and has analogous properties. For example, we have the following

**Theorem 1** *(Mean Value Theorem for scalar fields) Suppose $f$ is a scalar field. Assume $f'(a+tv;v)$ exists for all $0 \le t \le 1$. Then there is a $t_o$ with $0 \le t_o \le 1$ for which $f(a+v)-f(a) = f'(a+t_0v;v)$.*

**Proof**. Put $\phi(t) = f(a+tv)$. By hypothesis, $\phi$ is differentiable at every $t$ in $[0,1]$, and $\phi'(t) = f'(a+tv;v)$. By the one variable mean value theorem, there exists a $t_0$ such that $\phi'(t_0)$ is $\phi(1)-\phi(0)$, which equals $f(a+v)-f(a)$. Done.

When $v$ is a **unit vector**, $f'(a;v)$ is called the **directional derivative** of $f$ at $a$ in the direction of $v$.

The disadvantage of this construction is that it forces us to study the change of $f$ in one direction at a time. So we revisit the one-dimensional definition and note that the condition for differentiability there is equivalent to requiring that there exists a constant $c\,(=f'(a))$, such that $\lim\limits_{h\to 0}\left(\dfrac{f(a+h)-f(a)-ch}{h}\right) = 0$. If we put $L(h) = f'(a)h$, then $L:\mathbb{R}\to\mathbb{R}$ is clearly a linear map. We generalize this idea in higher dimensions as follows:

**Definition.** Let $f:\mathcal{D}\to\mathbb{R}^m$ ($\mathcal{D}\subseteq\mathbb{R}^n$) be a vector field and $a$ an interior point of $\mathcal{D}$. Then $f$ is differentiable at $x=a$ if and only if there exists a linear map $L:\mathbb{R}^n\to\mathbb{R}^m$ such that

$$(*) \qquad \lim_{u\to 0}\frac{||f(a+u)-f(a)-L(u)||}{||u||} = 0.$$

Note that the norm $||\cdot||$ denotes the length of vectors in $\mathbb{R}^m$ in the numerator and in $\mathbb{R}^n$ in the denominator. This should not lead to any confusion, however.

**Lemma 1** *Such an $L$, if it exists, is unique.*

**Proof.** Suppose we have $L, M:\mathbb{R}^n\to\mathbb{R}^m$ satisfying (*) at $x=a$. Then

$$\begin{aligned}\lim_{u\to 0}\frac{||L(u)-M(u)||}{||u||} &= \lim_{u\to 0}\frac{||L(u)+f(a)-f(a+u)+(f(a+u)-f(a)-M(u))||}{||u||}\\ &\le \lim_{u\to 0}\frac{||L(u)+f(a)-f(a+u)||}{||u||}\\ &\quad+\lim_{u\to 0}\frac{||f(a+u)-f(a)-M(u)||}{||u||} = 0.\end{aligned}$$

Pick any non-zero $v \in \mathbb{R}^n$, and set $u = tv$, with $t \in \mathbb{R}$. Then, the linearity of $L, M$ implies that $L(tv) = tL(v)$ and $M(tv) = tM(v)$. Consequently, we have

$$\begin{aligned} \lim_{t\to 0} \frac{||L(tv) - M(tv)||}{||tv||} &= 0 \\ &= \lim_{t\to 0} \frac{|t|\,||L(v) - M(v)||}{|t|\,||v||} \\ &= \frac{1}{||v||}||L(v) - M(v)||. \end{aligned}$$

Then $L(v) - M(v)$ must be zero.

**Definition.** If the limit condition $(*)$ holds for a linear map $L$, we call $L$ the **total derivative** of $f$ at $a$, and denote it by $T_a f$.

It is mind boggling at first to think of the derivative as a linear map. A natural question which arises immediately is to know what the value of $T_a f$ is at any vector $v$ in $\mathbb{R}^n$. We will show in section 2.3 that this value is precisely $f'(a; v)$, thus linking the two generalizations of the one-dimensional derivative.

Sometimes one can guess what the answer should be, and if (*) holds for this choice, then it must be the derivative by uniqueness. Here are **two examples** which illustrate this.

(1) Let $f$ be a **constant vector field**, i.e., there exists a vector $w \in \mathbb{R}^m$ such that $f(x) = w$, for all $x$ in the domain $\mathcal{D}$. Then we claim that $f$ is differentiable at any $a \in \mathcal{D}^0$ with **derivative zero**. Indeed, if we put $L(u) = 0$, for any $u \in \mathbb{R}^n$, then (*) is satisfied, because $f(a + u) - f(a) = w - w = 0$.

(2) Let $f$ be a **linear map**. Then we claim that $f$ is differentiable everywhere with $T_a f = f$. Indeed, if we put $L(u) = f(u)$, then by the linearity of $f$, $f(a + u) - f(a) = f(u)$, and so $f(a+u) - f(a) - L(u)$ is zero for any $u \in \mathbb{R}^n$. Hence (*) holds trivially for this choice of $L$.

Before we leave this section, it will be useful to take note of the following:

**Lemma 2** *Let $f_1, \ldots, f_m$ be the component (scalar) fields of $f$. Then $f$ is differentiable at $a$ iff each $f_i$ is differentiable at $a$. Moreover, $Tf(v) = (Tf_1(v), Tf_2(v), \ldots, Tf_n(g))$.*

An easy consequence of this lemma is that, when $\mathbf{n} = \mathbf{1}$, $f$ is differentiable at $a$ iff the following familiar looking limit exists in $\mathbb{R}^m$:

$$\lim_{h\to 0} \frac{f(a + h) - f(a)}{h},$$

allowing us to suggestively write $f'(a)$ instead of $T_a f$. Clearly, $f'(a)$ is given by the vector $(f_1'(a), \ldots, f_m'(a))$, so that $(T_a f)(h) = f'(a)h$, for any $h \in \mathbb{R}$.

**Proof**. Let $f$ be differentiable at $a$. For each $v \in \mathbb{R}^n$, write $L_i(v)$ for the $i$-th component of $(T_a f)(v)$. Then $L_i$ is clearly linear. Since $f_i(a+u) - f_i(u) - L_i(u)$ is the $i$-th component of $f(a+u) - f(a) - L(u)$, the norm of the former is less than or equal to that of the latter. This shows that (*) holds with $f$ replaced by $f_i$ and $L$ replaced by $L_i$. So $f_i$ is differentiable for any $i$. Conversely, suppose each $f_i$ differentiable. Put $L(v) = ((T_a f_1)(v), \ldots, (T_a f_m)(v))$. Then $L$ is a linear map, and by the triangle inequality,

$$||f(a+u) - f(a) - L(u)|| \leq \sum_{i=1}^{m} |f_i(a+u) - f_i(a) - (T_a f_i)(u)|.$$

It follows easily that (*) exists and so $f$ is differentiable at $a$.

## 2.2 Partial Derivatives

Let $\{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{R}^n$. The directional derivatives along the unit vectors $e_j$ are of special importance.

**Definition.** Let $j \leq n$. The $j$th partial derivative of $f$ at $x = a$ is $f'(a; e_j)$, denoted by $\frac{\partial f}{\partial x_j}(a)$ or $D_j f(a)$.

Just as in the case of the total derivative, it can be shown that $\frac{\partial f}{\partial x_j}(a)$ exists iff $\frac{\partial f_i}{\partial x_j}(a)$ exists for each coordinate field $f_i$.

**Example**: Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$f(x, y, z) = (e^{x\sin(y)}, z\cos(y)).$$

All the partial derivatives exist at any $a = (x_0, y_0, z_0)$. We will show this for $\frac{\partial f}{\partial y}$ and leave it to the reader to check the remaining cases. Note that

$$\frac{1}{h}(f(a + he_2) - f(a)) = \left(\frac{e^{x_0\sin(y_0+h)} - e^{x_0\sin(y_0)}}{h}, z_0\frac{\cos(y_0+h) - \cos(y_0)}{h}\right).$$

We have to understand the limit as $h$ goes to 0. Then the methods of one variable calculus show that the right hand side tends to the finite limit $(x_0\cos(y_0)e^{x_0\sin(y_0)}, -z_0\sin(y_0))$, which

is $\dfrac{\partial f}{\partial y}(a)$. In effect, the partial derivative with respect to $y$ is calculated like a one variable derivative, keeping $x$ and $z$ fixed. Let us note without proof that $\dfrac{\partial f}{\partial x}(a)$ is $(\sin(y_0)e^{x_0 \sin(y_0)}, 0)$ and $\dfrac{\partial f}{\partial z}(a)$ is $(0, \cos(y_0))$.

It is easy to see from the definition that $f'(a; tv)$ equals $tf'(a; v)$, for any $t \in \mathbb{R}$. This follows as $\frac{1}{h}(f(a + h(tv)) - f(a)) = t\frac{1}{th}(f(a + (ht)v) - f(a))$. In particular the Mean Value Theorem for scalar fields gives $f_i(a + hv) - f(a) = hf_i'(a + t_0hv) = hf_i(a + \tau v)$ for some $0 \leq \tau \leq h$.

We also have the following

**Lemma 3** *Suppose the derivatives of $f$ along any $v \in \mathbb{R}^n$ exist near $a$ and are continuous at $a$. Then*

$$f'(a; v + v') = f'(a; v) + f'(a; v'),$$

*for all $v, v'$ in $\mathbb{R}^n$. In particular, the directional derivatives of $f$ are all determined by the $n$ partial derivatives.*

We will do this for the scalar fields $f_i$. Notice

$$\begin{aligned} f_i(a + hv + hv') - f_i(a) &= f_i(a + hv + hv') - f_i(a + hv) + f_i(a + hv) - f(a) \\ &= hf_i(a + hv + \tau v') + hf_i(a + \tau' v) \end{aligned}$$

where here $0 \leq \tau \leq h$ and $0 \leq \tau' \leq h$. Now dividing by $h$ and taking the limit and $h \to 0$ gives $f_i'(a; v + v')$ for the first expression. The last expression gives a sum of two limits

$$\lim_{h \to 0} f_i'(a + hv + \tau v') + \lim_{h \to 0} f_i'(a + \tau' v; v').$$

But this is $f_i'(a; v) + f_i'(a; v')$.Recall both $\tau$ and $\tau'$ are between 0 and $h$ and so as $h$ goes to 0 so do $\tau$ and $\tau'$. Here we have used the continuity of the derivatives of $f$ along any line in a neighborhood of $a$.

Now pick $e_1, e_2, \ldots, e_n$ the usual orthogonal basis and recall $v = \sum \alpha_i e_i$. Then $f'(a; v) = f'(a; \sum \alpha_i e_i) = \sum \alpha_i f'(a; e_i)$. Also the $f'(a; e_i)$ are the partial derivatives. The Lemma now follows easily.

In the next section (Theorem $1a$) we will show that the conclusion of this lemma remains valid without the continuity hypothesis **if** we assume instead that $f$ has a total derivative at $a$.

The **gradient** of a scalar field $g$ at an interior point $a$ of its domain in $\mathbb{R}^n$ is defined to be the following vector in $\mathbb{R}^n$:

$$\nabla g(a) \;=\; grad\, g(a) \;=\; \left(\frac{\partial g}{\partial x_1}(a), \ldots, \frac{\partial g}{\partial x_n}(a)\right),$$

assuming that the partial derivatives exist at $a$.

Given a vector field $f$ as above, we can then put together the gradients of its component fields $f_i$, $1 \leq i \leq m$, and form the following important matrix, called the **Jacobian matrix** at $a$:

$$Df(a) \;=\; \left(\frac{\partial f_i}{\partial x_j}(a)\right)_{1\leq i\leq m, 1\leq j\leq n} \;\in\; M_{mn}(\mathbb{R}).$$

The $i$-th row is given by $\nabla f_i(a)$, while the $j$-th column is given by $\dfrac{\partial f}{\partial x_j}(a)$. Here we are using the notation $M_{mn}(\mathbb{R})$ for the collection of all $m \times n$-matrices with real coefficients. When $m = n$, we will simply write $M_n(\mathbb{R})$.

## 2.3 The main theorem

In this section we collect the main properties of the total and partial derivatives.

**Theorem 2** *Let $f : \mathcal{D} \to \mathbb{R}^m$ be a vector field, and $a$ an interior point of its domain $\mathcal{D} \subseteq \mathbb{R}^n$.*

**(a)** *If $f$ is differentiable at $a$, then for any vector $v$ in $\mathbb{R}^n$,*

$$(T_a f)(v) = f'(a, v).$$

*In particular, since $T_a f$ is linear, we have*

$$f'(a; \alpha v + \beta v') \;=\; \alpha f'(a; v) + \beta f'(a; v'),$$

*for all $v, v'$ in $\mathbb{R}^n$ and $\alpha, \beta$ in $\mathbb{R}$.*

**(b)** *Again assume that $f$ is differentiable. Then the matrix of the linear map $T_a f$ relative to the standard bases of $\mathbb{R}^n$, $\mathbb{R}^m$ is simply the Jacobian matrix of $f$ at $a$.*

**(c)** *$f$ differentiable at $a$ $\;\Rightarrow\;$ $f$ continuous at $a$.*

**(d)** *Suppose all the partial derivatives of $f$ exist near $a$ and are continuous at $a$. Then $T_a f$ exists.*

**(e)** *(chain rule) Consider*

$$\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m & \xrightarrow{g} & \mathbb{R}^h. \\ a & \mapsto & b = f(a) & & \end{array}$$

*Suppose $f$ is differentiable at $a$ and $g$ is diffentiable at $b = f(a)$. Then the composite function $h = g \circ f$ is differentiable at $a$, and moreover,*

$$T_a h \;=\; T_b g \circ T_a f.$$

*In terms of the Jacobian matrices, this reads as*

$$Dh(a) \;=\; Dg(b) Df(a) \;\in\; M_{kn}.$$

**(f)** $(m = 1)$ *Let $f$, $g$ be scalar fields, differentiable at $a$. Then*

*(i)* $T_a(f + g) = T_a f + T_a g$ *(additivity)*

*(ii)* $T_a(fg) = f(a)T_a g + g(a)T_a f$ *(product rule)*

*(iii)* $T_a(\frac{f}{g}) = \dfrac{g(a)T_a f - f(a)T_a g}{g(a)^2}$ *if* $g(a) \neq 0$ *(quotient rule)*

The following corollary is an immediate consequence of the theorem, which we will make use of, in the next chapter on normal vectors and extrema.

**Corollary 1** *Let $g$ be a scalar field, differentiable at an interior point $b$ of its domain $\mathcal{D}$ in $\mathbb{R}^n$, and let $v$ be any vector in $\mathbb{R}^n$. Then we have*

$$\nabla g(b) \cdot v \;=\; g'(b; v).$$

*Furthermore, let $\phi$ be a function from a subset of $\mathbb{R}$ into $\mathcal{D} \subseteq \mathbb{R}^n$, differentiable at an interior point $a$ mapping to $b$. Put $h = g \circ \phi$. Then $h$ is differentiable at $a$ with*

$$h'(a) \;=\; \nabla g(b) \cdot \phi'(a).$$

**Proof of main theorem**. (a) It suffices to show that $(T_a f_i)(v) = f_i(a; v)$ for each $i \leq n$. By definition,

$$\lim_{u \to 0} \frac{||f_i(a + u) - f_i(a) - (T_a f_i)(u)||}{||u||} = 0$$

This means that we can write for $u = hv$, $h \in \mathbb{R}$,

$$\lim_{h\to 0} \frac{f_i(a+hv) - f_i(a) - h(T_a f_i)(v)}{|h|||v||} = 0.$$

In other words, the limit $\lim\limits_{h\to 0} \frac{f_i(a+hv)-f_i(a)}{h}$ exists and equals $(T_a f_i)(v)$. Done.

(b) By part (a), each partial derivative exists at $a$ (since $f$ is assumed to be differentiable at $a$). The matrix of the linear map $T_a f$ is determined by the effect on the standard basis vectors. Let $\{e_i' | 1 \le i \le m\}$ denote the standard basis in $\mathbb{R}^m$. Then we have, by definition,

$$(T_a f)(e_j) = \sum_{i=1}^{m} (T_a f_i)(e_j) e_i' = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(a) e_i'.$$

The matrix obtained is easily seen to be $Df(a)$.

(c) First we need the following simple

**Lemma 4** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then, $\exists c > 0$ such that $||Tv|| \le c||v||$ for any $v \in \mathbb{R}^n$.*

**Proof of Lemma**. Let $A$ be the matrix of $T$ relative to the standard bases. Put $C = \max_j\{||T(e_j)||\}$. If $v = \sum\limits_{j=1}^{n} \alpha_j e_j$, then

$$\begin{aligned} ||T(v)|| &= ||\sum_j \alpha_j T(e_j)|| \le C \sum_{j=1}^{n} |\alpha_j| \cdot 1 \\ &\le C(\sum_{j=1}^{n} |\alpha_j|^2)^{1/2} (\sum_{j=1}^{n} 1)^{1/2} \le C\sqrt{n}||v||, \end{aligned}$$

by the Cauchy–Schwarz inequality. We are done by setting $c = C\sqrt{n}$.

This shows that a linear map is continuous as if $||v - w|| < \delta$ then $||T(v) - T(w)|| = ||T(v - w)|| < c||v - w|| < c\delta$.

(c) Suppose $f$ is differentiable at $a$. This certainly implies that the limit of the function $f(a + u) - f(a) - (T_a f)(u)$, as $u$ tends to $0 \in \mathbb{R}^n$, is $0 \in \mathbb{R}^m$ (from the very definition of $T_a f$, $||f(a + u) - f(a) - (T_a f)(u)||$ tends to zero "faster" than $||u||$, in particular it tends to zero). Since $T_a f$ is linear, $T_a f$ is continuous (everywhere), so that $\lim_{u\to 0}(T_a f)(u) = 0$. Hence $\lim_{u\to 0} f(a + u) = f(a)$ which means that $f$ is continuous at $a$.

(d) By hypothesis, all the partial derivatives exist near $a = (a_1, \ldots, a_n)$ and are continuous there. It suffices to show that each $f_i$ is differentiable at $a$ by lemma 2. So we have only to show that (*) holds with $f$ replaced by $f_i$ and $L(u) = f_i'(a; u)$. Write $u = (h_1, \ldots, h_n)$. By Lemma 3, we know that $f_i'(a; -)$ is linear. So

$$L(u) = \sum_{j=1}^{n} h_j \frac{\partial f_i}{\partial x_j}(a),$$

and we can write

$$f_i(a+u) - f_i(a) = \sum_{j=1}^{n} (\phi_j(a_j + h_j) - \phi_j(a_j)),$$

where each $\phi_j$ is a one variable function defined by

$$\phi_j(t) = f_i(a_1 + h_1, \ldots, a_{j-1} + h_{j-1}, t, a_{j+1}, \ldots, a_n).$$

By the mean value theorem,

$$\phi_j(a_j + h_j) - \phi_j(a_j) = h_j \phi_j'(t_j) = h_j \frac{\partial f_i}{\partial x_j}(y(j)),$$

for some $t_j \in [a_j, a_j + h_j]$, with

$$y(j) = (a_1 + h_1, \ldots, a_{j-1} + h_{j-1}, t_j, a_{j+1}, \ldots, a_n).$$

Putting these together, we see that it suffices to show that the following limit is zero:

$$\lim_{u \to 0} \frac{1}{||u||} |\sum_{j=1}^{n} h_j \left( \frac{\partial f_i}{\partial x_j}(a) - \frac{\partial f_i}{\partial x_j}(y(j)) \right)|.$$

Clearly, $|h_j| \leq ||u||$, for each $j$. So it follows, by the triangle inequality, that this limit is bounded above by the sum over $j$ of $\lim_{h_j \to 0} |\frac{\partial f_i}{\partial x_j}(a) - \frac{\partial f_i}{\partial x_j}(y(j))|$, which is zero by the continuity of the partial derivatives at $a$. Here we are using the fact that each $y(j)$ approaches $a$ as $h_j$ goes to 0. Done.

**Proof of (e)** Write $L = T_a f$, $M = T_b g$, $N = M \circ L$. *To show:* $T_a h = N$.

Define $F(x) = f(x) - f(a) - L(x-a)$, $G(y) = g(y) - g(b) - M(y-b)$ and $H(x) = h(x) - h(a) - N(x-a)$. Then we have

$$\lim_{x \to a} \frac{||F(x)||}{||x-a||} = 0 = \lim_{y \to b} \frac{||G(y)||}{||y-b||}.$$

So we need to show:

$$\lim_{x \to a} \frac{||H(x)||}{||x-a||} = 0.$$

But

$$H(x) = g(f(x)) - g(b) - M(L(x-a))$$

Since $L(x-a) = f(x) - f(a) - F(x)$, we get

$$H(x) = [g(f(x)) - g(b) - M(f(x) - f(a))] + M(F(x)) = G(f(x)) + M(F(x)).$$

Therefore it suffices to prove:

(i) $\lim\limits_{x \to a} \dfrac{||G(f(x))||}{||x-a||} = 0$ and

(ii) $\lim\limits_{x \to a} \dfrac{||M(F(x))||}{||x-a||} = 0.$

By Lemma 4, we have $||M(F(x))|| \leq c||F(x)||$, for some $c > 0$. Then $\dfrac{||M(F(x))||}{||x-a||} \leq c \lim\limits_{x \to a} \dfrac{||F(x)||}{||x-a||} = 0$, yielding (ii).

On the other hand, we know $\lim\limits_{y \to b} \dfrac{||G(y)||}{||y-b||} = 0$. So we can find, for every $\epsilon > 0$, a $\delta > 0$ such that $||G(f(x))|| < \epsilon||f(x) - b||$ if $||f(x) - b|| < \delta$. But since $f$ is continuous, $||f(x) - b|| < \delta$ whenever $||x - a|| < \delta_1$, for a small enough $\delta_1 > 0$. Hence

$$\begin{aligned} ||G(f(x))|| &< \epsilon||f(x) - b|| = \epsilon||F(x) + L(x-a)|| \\ &\leq \epsilon||F(x)|| + \epsilon||L(x-a)||, \end{aligned}$$

by the triangle inequality. Since $\lim\limits_{x \to a} \frac{||F(x)||}{||x-a||}$ is zero, we get

$$\lim_{x \to a} \frac{||G(f(x))||}{||x-a||} \leq \epsilon \lim_{x \to a} \frac{||L(x-a)||}{||x-a||}.$$

Applying Lemma 4 again, we get $||L(x-a)|| \leq c'||x-a||$, for some $c' > 0$. Now (i) follows easily.

(f) (i) We can think of $f+g$ as the composite $h = s(f,g)$ where $(f,g)(x) = (f(x), g(x))$ and $s(u,v) = u + v$ ("sum"). Set $b = (f(a), g(a))$. Applying (e), we get

$$T_a(f+g) = T_b(s) \circ T_a(f,g) = T_a(f) + T_a(g).$$

Done. The proofs of (ii) and (iii) are similar and will be left to the reader.

*QED.*

**Remark**. It is important to take note of the fact that a vector field $f$ may be differentiable at $a$ without the partial derivatives being continuous. We have a counterexample already when $n = m = 1$ as seen by taking

$$f(x) = x^2 \sin\left(\frac{1}{x}\right) \quad \text{if} \quad x \neq 0,$$

and $f(0) = 0$. This is differentiable everywhere. The only question is at $x = 0$, where the relevant limit $\lim\limits_{h\to 0} \frac{f(h)}{h}$ is clearly zero, so that $f'(0) = 0$. But for $x \neq 0$, we have by the product rule,

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right),$$

which does not tend to $f'(0) = 0$ as $x$ goes to 0. So $f'$ is not continuous at 0.

## 2.4 Mixed partial derivatives

Let $f$ be a scalar field, and $a$ an interior point in its domain $\mathcal{D} \subseteq \mathbb{R}^n$. For $j, k \leq n$, we may consider the second partial derivative

$$\frac{\partial^2 f}{\partial x_j \partial x_k}(a) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right)(a),$$

when it exists. It is called the *mixed partial derivative* when $j \neq k$, in which case it is of interest to know whether we have the equality

$$\frac{\partial^2 f}{\partial x_j \partial x_k}(a) = \frac{\partial^2 f}{\partial x_k \partial x_j}(a). \tag{3.4.1}$$

**Proposition 1** *Suppose* $\frac{\partial^2 f}{\partial x_j \partial x_k}$ *and* $\frac{\partial^2 f}{\partial x_k \partial x_j}$ *both exist near a and are continuous there. Then the equality (3.4.1) holds.*

The proof is similar to the proof of part (d) of Theorem 1.

# Chapter 3

# Tangent spaces, normals and extrema

If $S$ is a surface in 3-space, with a point $a \in S$ where $S$ looks *smooth*, i.e., without any *fold* or *cusp* or *self-crossing*, we can intuitively define the *tangent plane* to $S$ at $a$ as follows. Consider a plane $\Pi$ which lies outside $S$ and bring it closer and closer to $S$ till it touches $S$ near $a$ at only one point, namely $a$, without crossing into $S$. This intuitive picture is even clearer in the case of finding the *tangent line* to a smooth curve in the plane at a point. A slightly different way is to start with a plane $\Pi$ in $\mathbb{R}^3$ which slices $S$ in a small neighborhood $U = B_a(r)$ of $a$, and then consider the limit when $\Pi \cap U$ shrinks to the point $a$, and call the limiting plane the tangent plane at $a$. In the case of a plane curve arising as the graph of a nice function $f$, one knows that this process works and gives the slope of the tangent line to be $f'(a)$. In higher dimensions, if $S$ lies in $\mathbb{R}^n$, one can define tangent vectors by considering smooth curves on $S$ through the point $a$, and when $S$ is a *level set* of the form $f(x_1, \ldots, x_n) = c$, one can use the *gradient* of $f$ at $a$ to define the tangent space. But this method fails when the gradient vanishes at $a$. Finally, it should be noted that one often defines the *tangent space* at $a$, when it makes sense, in such a way that it becomes a vector space. But if one looks at the case of plane curves, the usual tangent line is not a vector space as it may not contain the origin $0$ $(= (0,0))$, unless of course if $a = 0$. It becomes a vector space if we *parallel translate* it to the origin by subtracting $a$. It is similar in higher dimensions. One calls a plane in 3-space an *affine plane* if it does not pass through the origin.

## 3.1 Tangents to parametrized curves

By a **parametrized curve**, or simply a *curve*, in $\mathbb{R}^n$, we mean the image $C$ of a continuous function

$$\alpha : [r, s] \to \mathbb{R}^n,$$

where $[r, s]$ is a closed interval in $\mathbb{R}$.

$C$ is called a **plane curve**, resp. a **space curve**, if $n = 2$, resp. $n = 3$. An example of the former, resp.the latter, is the **cycloid**, resp. the **right circular helix**, parametrized by $\alpha : [0, 2\pi] \to \mathbb{R}^2$ with $\alpha(t) = (t - \sin t, 1 - \cos t)$, resp. $\beta : [-3\pi, 6\pi] \to \mathbb{R}^3$ with $\beta(t) = (\cos t, \sin t, t)$.

Note that a parametrized curve $C$ has an **orientation**, i.e., a **direction**; it starts at $P = \alpha(r)$, moves along as $t$ increases from $r$, and ends at $Q = \alpha(s)$. We call the direction from $P$ to $Q$ positive and the one from $Q$ to $P$ negative. It is customary to say that $C$ is a **closed curve** if $P = Q$.

We say that $C$ is **differentiable** iff $\alpha$ is a differentiable function.

**Definition.** Let $C$ be a differentiable curve in $\mathbb{R}^n$ parametrized by an $\alpha$ as above. Let $a = \alpha(t_0)$, with $t_0 \in (r, s)$. Then $\alpha'(t_0)$ is called the **tangent vector** to $C$ at $a$ (in the positive direction).

For example, consider the unit circle $C$ in the plane given by $\alpha : [0, 2\pi] \to \mathbb{R}^2$, $t \to (\cos t, \sin t)$. Clearly $\alpha$ is differentiable, and we have $\alpha'(t) = (-\sin t, \cos t)$ for all $t \in (0, 2\pi)$. Consider the point $a = (1/2, \sqrt{3}/2)$ on the circle; clearly, $a = \alpha(\pi/3)$. By the above definition, the tangent vector to $C$ at $a$ is given by $\alpha'(\pi/3) = (-\sqrt{3}/2, 1/2)$. Let us now check that this agrees with our notion of a tangent from one variable Calculus. To this end we want to think of $C$ as the graph of a one variable function, but this is not possible for all of $C$. However, we can express the upper and the lower semi-circles as graphs of such functions. And since the point of interest $a$ lies on the upper semi-circle $C^+$, we look at the function $y = f(x)$ with $f(x) = \sqrt{1 - x^2}$, whose graph is $C^+$. Note that (by the chain rule) $f'(x) = -x(1 - x^2)^{-1/2}$. The **slope of the tangent** to $C^+$ at $a$ is then given by $f'(1/2) = -1/\sqrt{3}$. Hence the tangent line to $C$ at $a$ is the line $L$ passing through $(1/2, \sqrt{3}/2)$ of slope $-1/\sqrt{3}$. This is the unique line passing through $a$ in the direction of the tangent vector $(-\sqrt{3}/2, 1/2)$. (In other words $L$ is the unique line passing through $a$ and parallel to the line joining the origin to $(-\sqrt{3}/2, 1/2)$.) The situation is similar for any plane curve which is the graph of a one-variable function $f$, i.e. of the form $\alpha(t) = (t, f(t))$. So the definition of a tangent vector above is very reasonable.

## 3.2 Tangent Spaces and Normals to Level Sets.

In the last paragraph we have defined the tangent line to a parametrized curve. This is a set defined by a map *into* $\mathbb{R}^n$. Here we look at different kinds of sets which are defined by maps *from* $\mathbb{R}^n$.

**Definition.** Let $f : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \subset \mathbb{R}^n$, be a scalar field. For each $c \in \mathbb{R}$, the **level set** of $f$ at $c$ is given by

$$L_c = \{x \in \mathcal{D} \mid f(x) = c\}$$

**Example.** $f(x)$ = height from sea level, where $x$ is a point on Mount Wilson (sitting in $\mathbb{R}^3$). $L_c(f)$ consists of all points of constant height above sea level, just like those lines on a topographic map.

**Definition.** Let $f : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \subset \mathbb{R}^n$, be a differentiable scalar field and $a \in L_c(f)$. If $\nabla f(a) \neq 0$ then we define the **tangent space** $\Theta_a(L_c(f))$ to $L_c(f)$ at $a$ to be the vector space

$$\Theta_a(L_c(f)) = \{x \in \mathbb{R}^n \mid \nabla f(a) \cdot x = 0\}.$$

This is a solution set of one nonzero equation in $\mathbb{R}^n$, hence a vector space of dimension $n - 1$. One calls a point $a \in L_c(f)$ where $\nabla f(a) \neq 0$ a **smooth** point of $L_c(f)$. The idea being that a small neighborhood of $a$ in $L_c(f)$ looks just like a small neighborhod of any point in $\mathbb{R}^{n-1}$ though we won't make this statement precise.

**Definition.** Let $f$, $L_c(f)$ and $\Theta_a(L_c(f))$ be as in the previous definition. A **normal vector** (to $L_c(f)$ at $a$) is a vector $v \in \mathbb{R}^n$ orthogonal to all vectors in $\Theta_a(L_c(f))$.

It is clear that the normal vectors are just the scalar multiples of $\nabla f(a)$ (verify this!). Tangent vectors point "in the direction of" the level set whereas normal vectors point "as far away as possible" from the level set.

Our two notions of tangent space are related by the following observation.

**Proposition.** Let $f : \mathcal{D} \to \mathbb{R}$ be a differentiable scalar field, with $\mathcal{D} \subset \mathbb{R}^n$, $\alpha : \mathbb{R} \to L_c(f)$ a curve which is differentiable at $t_0 \in \mathbb{R}$ and so that $a = \alpha(t_0)$ is a smooth point of $L_c(f)$. Then $\alpha'(t_0) \in \Theta_a(L_c(f))$.

*Proof.* Since $\alpha$ takes values in the level set we have $f(\alpha(t)) = c$ for all $t \in \mathbb{R}$. Applying the chain rule to $h(t) := f(\alpha(t))$ (a constant function!) we find

$$0 = h'(t) \;=\; \nabla f(a) \cdot \alpha'(t_0)$$

which means that $\alpha'(t_0) \in \Theta_a(L_c(f))$. QED

In particular, if $n = 2$ we know that $\Theta_a(L_c(f))$ has dimension 1. So it must coincide with the tangent line defined for parametrized curves in the previous section. One can in fact show that given a smooth point $a \in L_c(f)$ there always is a curve $\alpha : \mathbb{R} \to L_c(f)$ with $\alpha(0) = a$ **and** $\alpha'(0)$ any given tangent vector in $\Theta_a(L_c(f))$. But we won't prove this here.

Given any subset $S$ of $\mathbb{R}^n$, and a point $a \in S$, one may ask whether there is always a good notion of tangent space to $S$ at $a$. If one starts with the idea that $\alpha'(t_0)$ for any parametrized curve : $\alpha : \mathbb{R} \to S$ should be a tangent vector, the tangent vectors might not form a linear space. Consider the example $S = \{(x, y) \in \mathbb{R}^2 | xy = 0\}$ at $a = (0, 0)$. The only differentiable curves in $S$ through $a$ (with non-zero gradient at 0) lie on one of the coordinate axes, and the tangent vectors are those of the form $(0, a)$ or $(a, 0)$. If a set has a "corner" or "cusp" at $a \in S$ there might be no differentiable curve with $a = \alpha(t_0)$ and $\alpha'(t_0) \neq 0$. Consider $S = \{(x, y) \in \mathbb{R}^2 | xy = 0, x \geq 0, y \geq 0\}$ and $a = (0, 0)$.

However, the two classes of sets we have studied so far readily generalize and so does the notion of tangent space for them.

**Definition** We say that $S \subset \mathbb{R}^n$ can be **differentiably parametrized** around $a \in S$ if there is a *bijective*, i.e., one-to-one and onto, differentiable function $\alpha : \mathbb{R}^k \to S \subset \mathbb{R}^n$ with $\alpha(0) = a$ and so that the linear map $T_0\alpha$ has largest possible rank, namely $k$. The **tangent space** to $S$ at $a$ is the simply the image of $T_0\alpha$, a linear subspace of $\mathbb{R}^n$. In particular we must have $k \leq n$.

**Definition**. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a vector field with components $(f_1, ..., f_m)$. Then for $c = (c_1, .., c_m) \in \mathbb{R}^m$ the level set

$$L_c(f) := \{x \in \mathbb{R}^n | f(x) = c\} = L_{c_1}(f_1) \cap ... \cap L_{c_m}(f_m)$$

is defined as before, and the tangent space

$$\Theta_a(L_c(f)) := \{x \in \mathbb{R}^n | (T_a f)(x) = 0\}$$

is defined if $T_a f$ has largest possible rank, namely $m$. This latter condition is equivalent to requiring $\nabla f_1(a), ..., \nabla f_m(a)$ to be linearly independent. We also have

$$\Theta_a(L_c(f)) = \Theta_a(L_{c_1}(f_1)) \cap ... \cap \Theta_a(L_{c_m}(f_m))$$

and $\dim_{\mathbb{R}} \Theta_a(L_c(f)) = n - m$.

Once we have the notion of tangent space, a normal vector is defined as before for both classes of examples. Note that $n - m$ is also the (intuitive) dimension of $L_c(f)$.

**Examples**:

(1) Find a normal to the surface $S$ given by $z = x^2y^2 + y + 1$ at $(0, 0, 1)$: Set $f(x, y, z) = x^2y^2 + y - z + 1$. Then $z = x^2y^2 + y + 1$ iff $(x, y, z)$ lies on the level set $L_0(f)$. Since $f$ is a polynomial function, it is differentiable. We have:

$$\nabla f(a) = (2xy^2, 2x^2y + 1, -1)|_{(0,0,1)} = (0, 1, -1),$$

which is a normal to $L_0(f) = S$. Indeed, as the proof of the Proposition above shows, to be normal to a level set $L_c(f)$ at a point $a$ is the same as being perpendicular to the gradient of $f$ at that point, namely $\nabla f(a)$. So a unit normal is given by $n = (0, 1/\sqrt{2}, -1/\sqrt{2})$.

(2) (Physics example) Let $P$ be a unit mass particle at $(x, y, z)$ in $\mathbb{R}^3$. Denote by $\vec{F}$ the **gravitational force** on $P$ exerted by a point mass $M$ at the origin. Then Newton's gravitational law tells us that

$$\vec{F} = -\frac{GM}{r^2}\left(\frac{\vec{r}}{r}\right),$$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, $r = ||\vec{r}||$, and $G$ the gravitational constant. (Here $\vec{i}, \vec{j}, \vec{k}$ denote the standard basis vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.) Put $V = -GM/r$, which is called the **gravitational potential**. By chain rule, $\frac{\partial V}{\partial x}$ equals $\frac{GM}{r^2}\frac{\partial r}{\partial x}$. Similarly for $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$. Therefore

$$-\nabla V = \frac{GM}{r^2}\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = \frac{GM}{r^2}\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \vec{F}.$$

For $c < 0$, the level sets $L_c(V)$ are spheres centered at the origin of radius $-GM/c$, and $\vec{F}$ is evidently perpendicular to each of them. Note that $F$ gets stronger as $r$ gets smaller as the physical intuition would suggest.

(3) Let $f = x^3 + y^3 + z^3$ and $g = x + y + z$. The level set $S := L_{10}(f) \cap L_4(g)$ is a curve. The set $S \cap \{(x, y, z) \in \mathbb{R}^3 | x \geq 0, y \geq 0, z \geq 0\}$ is closed and bounded hence compact (verify this!).

## 3.3 Relative maxima, relative minima and saddle points.

Let $f : \mathcal{D} \to \mathbb{R}$ be a scalar field, and let $a$ be a point in the interior of $\mathcal{D}$.

**Definition**. The point $a = (a_1, \ldots, a_n)$ is a **relative maximum** (resp. **relative minimum**) iff we can find an open ball $B_a(r)$ completely contained in $\mathcal{D}$ such that for *every* $x$ in $B_a(r)$, $f(x) \leq f(a)$ (resp. $f(x) \geq f(a)$).

By a **relative extremum** we mean a point $a$ which is either a relative maximum or a relative minimum. Some authors call it a *local extremum.*

*Examples*:

(1) Fix real numbers $b_1, b_2, \ldots, b_n$, not all zero, and consider $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x_1, x_2, \ldots, x_n) = b_1x_1 + b_2x_2 + \ldots + b_nx_n$. In vector language, this is described by $f(x) = b \cdot x$. Then *no* point in $\mathbb{R}^n$ is a relative extremum for this function. More precisely, we claim that, given any $a$ in $\mathbb{R}^n$ and any other vector $v \in \mathbb{R}^n$, $f(x) - f(a)$ is positive and negative for all $x$ near $a$ along the direction defined by $v$. Indeed, if we write $x = a + tv$, the linearity of $f$ shows that $f(x) - f(a) = tf(v)$, which changes sign when $t$ does.

(2) Fix a non-zero real number $b$ and consider the scalar field $f(x, y) = b(x^2 + y^2)$ on $\mathbb{R}^2$. It is an easy exercise to see that $f$ has a local extremum at $(0, 0)$, which is a relative maximum iff $b$ is negative.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x - [x]$, where $[x]$ denotes the largest integer which is less than or equal to $x$, called the *integral part* of $x$. (For example, $[\pi] = 3$, $[\sqrt{2}] = 1$ and $[-\pi] = -4$.) Then it is easy to see that $f(x)$ is always non-negative and that $f(n) = 0$ for all integers $n$. Hence the integers $n$ are all relative minima for $f$. The graph of $f$ is continuous except at the integers and is periodic of period 1. Moreover, the image of $f$ is the half-closed interval $[0, 1)$. We claim that there is no relative maximum. Indeed the function is increasing (with slope 1) on $[n, n + 1)$ for any integer $n$, any relative maximum on $[n, n + 1)$ has to be an absolute maximum and this does not exist; $f(x)$ can get arbitrarily close to 1 but can never attain it.

In the first two examples above, $f$ was continuous, even differentiable. In such cases the relative extrema exert an influence on the derivative, which can be used to locate them.

**Lemma.** If $a = (a_1, \ldots, a_n)$ is a relative extremum of a differentiable scalar field $f$, then $\nabla f(a)$ is zero.

*Proof.* For every $j = 1, \ldots, n$, define a one-variable function $g_j$ by

$$g_j(t) = f(a_1, \ldots, a_{j-1}, a_j + t, a_{j+1}, \ldots, a_n),$$

so that $g_j(0) = f(a)$. Since $f$ has a relative extremum at $a$, $g_j$ has a local extremum at 0. Since $f$ is differentiable, $g_j$ is differentiable as well, and we see that by Ma 1a material, $g_j'(0) = 0$ for *every* $j$. But $g_j'(0)$ is none other than $\frac{\partial f}{\partial x_j}(a)$. Since $\nabla f(a) = (\frac{\partial f}{\partial x_1}(a), \ldots \frac{\partial f}{\partial x_n}(a))$, it must be zero.

QED.

**Definition.** A **stationary point** of a differentiable scalar field $f$ is a point $a$ where $\nabla f(a) = 0$. It is called a **saddle point** if it is not a relative extremum.

Some call a stationary point a *critical point*. In the language of the previous section, stationary points are the non-smooth points of the level set. But here we focus on the function rather than its level sets, hence the new terminology.

Note that if $a$ is a saddle point, then by definition, *every* open ball $B_a(r)$ contains points $x$ with $f(x) \geq f(a)$ and points $x'$ such that $f(x') \leq f(a)$.

*Examples* (contd.):

(3) Let $f(x, y) = x^2 - y^2$, for all $(x, y) \in \mathbb{R}^2$. Being a polynomial, $f$ is differentiable everywhere, and $\nabla f = (2x, -2y)$. So the origin is the unique stationary point. We claim that it is a saddle point. Indeed, consider any open ball $B_0(r)$ centered at 0. If we move away from 0 along the $y$ direction (inside $B_0(r)$), the function is $z = -y^2$ and so the origin is a local maximum along this direction. But if we move away from 0 in the $x$ direction, we see that the origin is a local minimum along that direction. So $f(x, y) - f(0, 0)$ is both positive and negative in $B_0(r)$, and this is true for any $r > 0$. Hence the claim. Note that if we graph $z = x^2 - y^2$ in $\mathbb{R}^3$, the picture looks quite a bit like a real saddle around the origin. This is the genesis of the mathematical terminology of a saddle point.

(4) The above example can be jazzed up to higher dimensions as follows: Fix positive integers $n, k$ with $k < n$, and define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x_1, x_2, \ldots, x_n) \;=\; x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_n^2.$$

Since $\nabla f = 2(x_1, \ldots, x_k, -x_{k+1}, \ldots, x_n)$, the origin is the only stationary point. Again, it is a saddle point by a similar argument.

## 3.4 The Hessian test for relative max/min

Given a *stationary point* $a$ of a nice two-variable function (*scalar field*) $f$, say with continuous second partial derivatives at $a$, there is a test one can apply to determine whether it is a *relative maximum* or a *relative minimum* or neither. This generalizes in a non-obvious way the second derivative test in the one-variable situation. One crucial difference is that in that (one-variable) case, one does not require the continuity of the second derivative at $a$. This added hypothesis is needed in our (2-variable) case because the second mixed partial derivatives, namely $\partial^2 f/\partial x \partial y$ and $\partial^2 f/\partial y \partial x$ need not be equal at $a$.

First we need to define the *Hessian*. Suppose we are given a function $f : \mathcal{D} \to \mathbb{R}$ with $\mathcal{D} \subset \mathbb{R}^n$, and an interior point $a$ of $\mathcal{D}$ where the second partial derivatives $\partial^2 f/\partial x_i \partial x_j$ exist for all $i, j \leq n$. We put

$$Hf(a) \;=\; \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right),$$

which is clearly an $n \times n$-matrix. It is a **symmetric matrix** iff all the mixed partial derivatives are independent of the order in which the partial differentiation is performed. The reason symmetry is of interest is that we know by a basic result in *Linear Algebra* (Ma 1b material) that any (real) symmetric matrix can be diagonalized with all eigenvalues being real. The **Hessian determinant** of $f$ at $a$ is by definition the determinant of $Hf(a)$, denoted $\det(Hf(a))$.

We will make use of this notion in the plane.

**Theorem** *Let $\mathcal{D} \subset \mathbb{R}^2$, $f : \mathcal{D} \to \mathbb{R}$ a scalar field, and $a \in \mathcal{D}^0$. Assume that all the second partial derivatives of $f$ exist and are continuous at $a$. Suppose $a$ is a stationary point. Then*

*(i) If $\partial^2 f/\partial x^2(a) < 0$ and det$(Hf(a)) > 0$, then $a$ is a relative maximum for $f$;*

*(ii) If $\partial^2 f/\partial x^2(a) > 0$ and det$(Hf(a)) > 0$, then $a$ is a relative maximum for $f$;*

*(iii) If det$(Hf(a)) < 0$, $a$ is a saddle point for $f$.*

The test does not say anything if $Hf(a)$ is *singular*, i.e., if it has zero determinant.

Since the second partial derivatives of $f$ exist and are continuous at $a$, we know from the previous chapter that

$$\frac{\partial^2 f}{\partial x \partial y}(a) \;=\; \frac{\partial^2 f}{\partial y \partial x}(a).$$

This means the matrix $Hf(a)$ is symmetric (and real). So by a theorem in *Linear Algebra* (Ma1b), we know that we can diagonalize it. In other words, we can find an invertible matrix $M$ such that

$$M^{-1} Hf(a) M \;=\; \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_1, \lambda_2$ are the *eigenvalues*. (We can even choose $M$ to be an *orthogonal* matrix, i.e., with $M^{-1}$ being the transpose matrix $M^t$.) Then we have

$$(D) \qquad \det(Hf(a)) = \lambda_1 \lambda_2,$$

and

$$(T) \qquad \mathrm{tr}(Hf(a)) = \partial^2 f/\partial x^2(a) + \partial^2 f/\partial y^2(a) \;=\; \lambda_1 + \lambda_2,$$

where tr denotes the *trace*. (Recall that the determinant and the trace do not change when a matrix is conjugated by another matrix, such as $M$.)

Now suppose $Hf(a)$ has positive determinant, so that

$$\partial^2 f/\partial x^2(a)\partial^2 f/\partial y^2(a) \;>\; \left(\partial^2 f/\partial x\partial y(a)\right)^2 \geq 0.$$

In this case, by (D), $\lambda_1$ and $\lambda_2$ have the same sign. Using (T), we then see that $\partial^2 f/\partial x^2(a)$ and $\partial^2 f/\partial y^2(a)$ must have the same sign as $\lambda_j$, $j = 1, 2$. When this sign is *positive* (resp. *negative*), $a$ is a *relative minimum* (resp. *relative maximum*) for $f$.

When the Hessian determinant is negative at $a$, $\lambda_1$ and $\lambda_2$ have opposite sign, which means that $f$ increases in some direction and decreases in another. In other words, $a$ is a saddle point for $f$.

## 3.5 Constrained extrema and Lagrange multipliers.

Very often in practice one needs to find the extrema of a scalar field subject to some constraints imposed by the geometry of the situation. More precisely, one starts with a scalar field $f : \mathcal{D} \to \mathbb{R}$, with $\mathcal{D} \subset \mathbb{R}^n$, and a subset $S$ of the interior of $\mathcal{D}$, and asks what the maxima and minima of $f$ are when restricted to $S$.

For example, we could consider the saddle function $f(x, y) = x^2 - y^2$ with $\mathcal{D} = \mathbb{R}^2$, which has no relative extrema, and ask for the constrained extrema relative to

$$S = \text{unit circle} \;:\quad x^2 + y^2 \;=\; 1$$

This can be solved explicitly. A good way to do it is to look at the level curves $L_c(f)$. For $c = 0$, we get the union of the lines $y = x$ and $y = -x$. For $c \neq 0$, we get hyperbolas with foci $(\pm\sqrt{c}, 0)$ if $c > 0$ and $(0, \pm\sqrt{-c})$ if $c < 0$. It is clear by inspection that the constrained extrema of $f$ on $S$ are at the four points $(1, 0), (-1, 0), (0, 1), (0, -1)$; the former two are constrained maxima (with value 1) and the latter two are constrained minima (with value $-1$). If you like a more analytical proof, see below.

The question of finding constrained extrema is very difficult to solve in general. But luckily, if $f$ is differentiable and if $S$ is given as the intersection of a finite number of level sets of differentiable functions (with independent gradients), there is a beautiful method due to Lagrange to find the extrema, which is efficient at least in low dimensions. Here is Lagrange's result.

**Theorem 1** *Let $f$ be a differentiable scalar field on $\mathbb{R}^n$. Suppose we are given scalars $c_1, c_2, \ldots, c_r$ and differentiable functions $g_1, g_2, \ldots, g_r$ on $\mathbb{R}^n$, so that the constraint set $S$ is the intersection of the level sets $L_{c_1}(g_1), L_{c_2}(g_2), \cdots, L_{c_r}(g_r)$. Let $a$ be a point on $S$ where $\nabla g_1(a), \nabla g_2(a), \cdots, \nabla g_r(a)$ are linearly independent. Then we have the following:*

*If $a$ is a constrained extremum of $f$ on $S$, then there are scalars $\lambda_1, \lambda_2, \cdots, \lambda_r$ (called the "Lagrange Multipliers") such that*

$$\nabla f(a) = \lambda_1 \nabla g_1(a) + \cdots + \lambda_r \nabla g_r(a).$$

**Proof**. We use the fact, mentioned for $r = 1$ in the previous section, that one can always find a differentiable curve $\alpha : \mathbb{R} \to S$ with $\alpha(0) = a$ and $\alpha'(0)$ any given tangent vector in $\Theta_a(S)$ (note that $S$ is a generalized level set and the linear independence of the $\nabla g_i(a)$ assures that $\Theta_a(S)$ is defined). This is not so easy to prove and we may do this later in the course if there is time left.

So if $a$ is a constrained extremum of $f$ on $S$ we take such a curve $\alpha$ and notice that $a$ is also an extremum of $f$ on the image of $\alpha$. So the one-variable function $h(t) := f(\alpha(t))$ has vanishing derivative at $t = 0$ which by the chain rule implies

$$\nabla f(a) \cdot \alpha'(0) = 0.$$

Since $\alpha'(0)$ was any given tangent vector we find that $\nabla f(a)$ is orthogonal to the whole tangent space (i.e. it is a normal vector). The normal space $N$, say, has dimension $r$ since it is a solution space of a system of linear equations of rank $n - r$ (picking a basis $\theta_1, ..., \theta_{n-r}$ of $\Theta_a(L_c(f))$ the system of equations is $\theta_i \cdot x = 0$). On the other hand $\nabla g_i(a)$ all lie in $N$ and they are linearly independent by assumption, so they form a basis of $N$. Since we've found that $\nabla f(a)$ lies in $N$, $\nabla f(a)$ must be a linear combination of these basis vectors. Done.

QED.

Now let us use (a superspecial case of) this Theorem to check the answer we gave above for the problem of finding constrained extrema of $f(x, y) = x^2 - y^2$ relative to the unit circle $S$ centered at $(0, 0)$. Here $r = 1$.

Note first that $S$ is the level set $L_0(g)$, where $g(x, y) = x^2 + y^2 - 1$. We need to solve $\nabla f = \lambda \nabla g$. Since $\nabla f = (2x, -2y)$ and $\nabla g = (2x, 2y)$, we are left to solve the simultaneous equations $x = \lambda x$ and $-y = \lambda y$. For this to happen, $x$ and $y$ cannot both be non-zero, as otherwise we will get $\lambda = 1$ from the first equation and $\lambda = -1$ from the second! They cannot both be zero either, as $(x, y)$ is constrained to lie on $S$ and so must satisfy $x^2 + y^2 = 1$. So, either $x = 0$, $y = \pm 1$, when $\lambda = -1$, or $x = \pm 1$, $y = 0$, in which case $\lambda = 1$. So the four candidates for constrained extrema are $(0, \pm 1), (\pm 1, 0)$. One can now check that they are indeed constrained extrema. We get the same answer as before.

We come back to the example of the set $S$ defined as the intersection of the two level sets $x^3+y^3+z^3=10$ and $x+y+z=4$. Suppose we want to find points on $S$ with maximal or minimal distance from the origin. We need to look at $f(x,y,z)=x^2+y^2+z^2$ and solve the system of five equations in five variables

$$\begin{aligned} x^3+y^3+z^3 =& 10 \\ x+y+z =& 4 \\ \nabla f(x,y,z)=(2x,2y,2z) =& \lambda_1(3x^2,3y^2,3z^2)+\lambda_2(1,1,1). \end{aligned}$$

These are nonlinear equations for which there is no simple method like the Gauss Jordan algorithm for linear equations. It's a good idea in every such problem to look at symmetries. Our set $S$ is invariant under any permutation of the coordinates $x,y,z$, so it has a sixfold symmetry. Each of $x,y,z$ satisfies the quadratic equation $2t=3\lambda_1 t^2+\lambda_2$ or

$$t^2-\frac{2}{3\lambda_1}t+\frac{\lambda_2}{3\lambda_1}=0. \qquad (*)$$

Let's check that $\lambda_1=0$ cannot happen. Because if $\lambda_1=0$ we'd have $x=y=z=\lambda_2/2$ and $x+y+z=3\lambda_2/2=4$ hence $\lambda_2=8/3$. But $x^3+y^3+z^3=3(4/3)^3\neq 10$. Contradiction! We've also shown here that $x=y=z$ cannot happen. In particular our quadratic equation (*) has two distinct roots, and as you well know it has exactly two roots! So two of $x,y,z$ must coincide: $x=y\neq z$, or $x=z\neq y$ or $x\neq y=z$. Let's look at the second case. Then $2x+y=4$ and $y=4-2x$ which leads to $2x^3+(4-2x)^3=10$. This is a cubic equation which, luckily, has an obvious solution, namely $x=1$ (if you were not so lucky there'd still be *some* solution in $\mathbb{R}$). Now $x=1$ leads to $(1,2,1)=\lambda_1(3,12,3)+\lambda_2(1,1,1)$ which we know how to solve (if there is a solution). We find $\lambda_1=1/9$ and $\lambda_2=2/3$ is a solution. The value of $f$ at $(1,2,1)$ is $1^2+2^2+1^2=6$.

Now the cubic $2x^3+(4-2x)^3-10=0$ written in standard form is $x^3-8x^2+16x-9$ and it factors as $(x-1)(x^2-7x+9)$. The two roots of the quadratic part are $\frac{7\pm\sqrt{13}}{2}$ and we check that for each of them we find corresponding $\lambda_1$, $\lambda_2$ simply by identifying $x^2-7x+9$ with $(*)$. Suppose now we look only at the part of $S$ where all of $x,y,z$ are positive. Then there is only the possibility $x_0:=\frac{7-\sqrt{13}}{2}, y_0:=4-2x=-3+\sqrt{13}, z_0:=\frac{7-\sqrt{13}}{2}$. And we have $f(x_0,y_0,z_0)=53-13\sqrt{13}\sim 6.12$. This is bigger than 6, so $(x_0,y_0,z_0)$ is a maximum and $(1,2,1)$ is a minimum (our function really doesn't vary all that much). Note that we know a priori that a maximum *and* a minimum must exist because the points in $S$ with positive coordinates form a closed and bounded, hence compact set.

Taking into account the possibilities where the other two coordinates coincide, we've found three maxima and three minima of $f$.

**Remark**. Suppose $f:\mathcal{D}\to\mathbb{R}$ is a differentiable scalar field, with $\mathcal{D}\subset\mathbb{R}$, and $S$ a subset of $\mathcal{D}$. Suppose $a$ is a constrained extremum on $S$. Then $a$ must be one of the following:

(1) $a$ is a relative extremum on $S$;

(2) $a$ is a boundary point on $S$.

**Gradients and parallelism**:

We know how the gradients of a differentiable scalar field $f$ give (non-zero) normal vectors to the level sets $L_c(f)$ at points $a$ where $\nabla f(a) \neq 0$. This is used in the following instances:

(1) To find points of tangency between the level sets $L_c(f)$ and $L_{c'}(g)$. In this case, if $a$ is a point of tangency, $\nabla f(a)$ and $\nabla g(a)$ must be proportional.

(2) To find the extrema with constraints using Lagrange multipliers.

(3) To determine if two lines in $\mathbb{R}^2$ meet. (*Solution*: Two lines $L_1$ and $L_2$ in $\mathbb{R}^2$ meet if and only if their gradients are not proportional.) Similarly, one may want to know if two planes in $\mathbb{R}^3$ meet, etc.

One can use the following approach to solve problem (3), which generalizes well to higher dimensions.

Let

$L_1$=level set of $f(x, y) = a_1 x + a_2 y$ at $-a_0$

$L_2$=level set of $g(x, y) = b_1 x + b_2 y$ at $-b_0$

Check if $\nabla f$ is proportional to $\nabla g$. ($\nabla f = (a_1, a_2)$ and $\nabla g = (b_1, b_2)$.)

$L_1$ and $L_2$ meet if and only if $(a_1, a_2) \neq \lambda(b_1, b_2)$. In other words, $\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ should not be zero.

There are 3 possibilities for $L_1 \cap L_2$:

(1) $L_1 \cap L_2 = \emptyset$. They are parallel but not equal, $\nabla f = \lambda \nabla g$ but $L_1 \neq L_2$

(2) $L_1 \cap L_2 = L_1 = L_2$: $L_1$ and $L_2$ are the same.

(3) $L_1 \cap L_2$ is a point: This happens IFF $\nabla f$ is not proportional to $\nabla g$ IFF $\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \neq 0$.

# CHAPTER 4 – MULTIPLE INTEGRALS

The objects of this chapter are five-fold. They are:

(1) Discuss when scalar-valued functions $f$ can be integrated over closed rectangular boxes $R$ in $\mathbb{R}^n$; simply put, $f$ is *integrable* over $R$ iff there is a unique real number, to be denoted $I_R(f)$ or $\int_R f$ when it exists, which is caught between the *upper and lower sums* relative to *any* partition $P$ of $R$.
(2) Show that any *continuous* function $f$ can be integrated over $R$.
(3) Discuss Fubini's theorem, which when applicable, allows one to do multiple integrals as *iterated integrals*, i.e, integrate one variable at a time.
(4) Show that bounded functions with *negligible sets of discontinuities* can be integrated over $R$.
(5) Discuss integrals of continuous functions over general compact sets.

## §4.1 Basic notions

We will first discuss the question of integrability of bounded functions on closed rectangular boxes, and then move on to integration over slightly more general regions.

Recall that in one variable calculus, the integral of a function over an interval $[a, b]$ was defined as the limit, when it exists, of certain sums over finite partitions $P$ of $[a, b]$ as $P$ becomes finer and finer. To try to transport this idea to higher dimensions, we need to generalize the notions of partition and refinement.

In this chapter, $R$ will always denote a closed rectangular box in $\mathbb{R}^n$, written as $[a, b] = [a_1, b_1] \times \cdots \times [a_n, b_n]$, where $a_j, b_j, \in \mathbb{R}$ for all $j$ with $a_j < b_j$.

**Definition.** A **partition** of $R$ is a finite collection $P$ of subrectangular (closed) boxes $S_1, S_2, \ldots, S_r \subseteq R$ such that

(i) $R = \cup_{j=1}^r S_j$, and
(ii) the interiors of $S_i$ and $S_j$ have no intersection for all $i \neq j$.

**Definition.** A **refinement** of a partition $P = \{S_j\}_{j=1}^r$ of $R$ is another partition $P' = \{S'_k\}_{k=1}^m$ with each $S'_k$ contained in some $S_j$.

It is clear from the definition that given any two partitions $P, P'$ of $R$, we can find a third partition $P''$ which is simultaneously a refinement of $P$ and of $P'$.

Now let $f$ be a bounded function on $R$, and let $P = \{S_j\}_{j=1}^r$ a partition of $R$. Then $f$ is certainly bounded on each $S_j$, i.e., $f(S_j)$ is a bounded subset of $\mathbb{R}$. It was proved in Chapter 2 that every bounded subset of $\mathbb{R}$ admits a **sup** (lowest upper bound) and an **inf** (greatest lower bound).

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**Definition.** The **upper** (resp. **lower**) **sum** of $f$ over $R$ relative to the partition $P = \{S_j\}_{j=1}^r$ is given by

$$U(f,P) = \sum_{j=1}^{r} \mathrm{vol}(S_j)\sup(f(S_j))$$
$$\left(\text{resp.}\quad L(f,P) = \sum_{j=1}^{r} \mathrm{vol}(S_j)\inf(f(S_j)).\right)$$

Here $\mathrm{vol}(S_j)$ denotes the volume of $S_j$. Of course, we have

$$L(f,P) \le U(f,P)$$

for all $P$.

More importantly, it is clear from the definition that if $P' = \{S'_k\}_{k=1}^m$ is a refinement of $P$, then

$$L(f,P) \le L(f,P') \quad \text{and} \quad U(f,P') \le U(f,P).$$

Put

$$\mathcal{L}(f) = \{L(f,P) \mid P \text{ partition of } R\} \subseteq \mathbb{R}$$

and

$$\mathcal{U}(f) = \{U(f,P) \mid P \text{ partition of } R\} \subseteq \mathbb{R}.$$

**Lemma.** *$\mathcal{L}(f)$ admits a* **sup***, denoted $\underline{I}(f)$, and $\mathcal{U}(f)$ admits an* **inf***, denoted $\overline{I}(f)$.*

*Proof.* Thanks to the discussion in Chapter 2, all we have to do is show that $\mathcal{L}(f)$ (resp. $\mathcal{U}(f)$) is bounded from above (resp. below). So we will be done if we show that given any two partitions $P, P'$ of $R$, we have $L(f,P) \le U(f,P')$ as then $\mathcal{L}(f)$ will have $U(f,P')$ as an upper bound and $\mathcal{U}(f)$ will have $L(f,P)$ as a lower bound. Choose a third partition $P''$ which refines $P$ and $P'$ simultaneously. Then we have $L(f,P) \le L(f,P'') \le U(f,P'') \le U(f,P')$. Done. $\square$

**Definition.** A bounded function $f$ on $R$ is **integrable** iff $\underline{I}(f) = \overline{I}(f)$. When such an equality holds, we will simply write $I(f)$ (or $I_R(f)$ if the dependence on $R$ needs to be stressed) for $\underline{I}(f)$ $(= \overline{I}(f))$, and call it the **integral of $f$ over $R$**. Sometimes we will write

$$I(f) = \int_R f \quad \text{or} \quad \int \cdots \int_R f(x_1, \dots, x_n)\, dx_1 \dots dx_n.$$

Clearly, when $n = 1$, we get the integral we are familiar with, often written as $\int_{a_1}^{b_1} f(x_1)\, dx_1$.

This definition is hard to understand, and a useful criterion is given by the following

**Lemma.** *$f$ is integrable over $R$ iff for every $\varepsilon > 0$ we can find a partition $P$ of $R$ such that*

$$U(f,P) - L(f,P) \, < \, \varepsilon.$$

*Proof.* If $f$ is integrable, $I_R(f)$ is arbitrarily close to the sets of upper and lower sums, and we can certainly find, given any $\varepsilon > 0$, some $P$ such that $I_R(f) - L(f,P) < \varepsilon/2$ and $U(f,P) - I_R(f) < \varepsilon/2$. Done in this direction. In the converse direction, since

$$L(f,P) \le \underline{I}(f) \le \overline{I}(f) \le U(f,P)$$

for any $P$, if $U(f,P) - L(f,P) < \varepsilon$, we must have

$$\overline{I}(f) - \underline{I}(f) < \varepsilon.$$

Since $\varepsilon$ is an arbitrary positive number, $\underline{I}(f)$ must equal $\overline{I}(f)$, i.e., $f$ is integrable over $R$.

## §4.2 Step functions

The obvious question now is to ask if there are integrable functions. One such example is given by the **constant function** $f(x) = c$, for all $x \in R$. Then for any partition $P = \{S_j\}$, we have

$$L(f,P) = U(f,P) = c\sum_{j=1}^{r} \mathrm{vol}(S_j) = c \ \mathrm{vol}(R).$$

So $\underline{I}(f) = \overline{I}(f)$ and $\int_R f = c \ \mathrm{vol}(R)$.

This can be jazzed up as follows.

**Definition.** A **step function on $\boldsymbol{R}$** is a function $f$ on $R$ which is constant on each of the subrectangular boxes $S_j$ of **some** partition $P$.

**Lemma.** *Every step function $f$ on $R$ is integrable.*

*Proof.* By definition, there exists a partition $P = \{S_j\}_{j=1}^r$ of $R$ and scalars $\{c_j\}$ such that $f(x) = c_j$, if $x \in S_j$. Then, arguing as above, it is clear that for **any** refinement $P'$ of $P$, we have

$$L(f,P') = U(f,P') = \sum_{j=1}^{r} c_j \ \mathrm{vol}(S_j).$$

Hence, $\underline{I}(f) = \overline{I}(f)$. $\square$

## §4.3 Integrability of continuous functions

The most important bounded functions on $R$ are continuous functions. (Recall from Chapter 1 that every continuous function on a compact set is bounded, and that $R$ is compact.) The first result of this chapter is given by the following

**Theorem.** *Every continuous function $f$ on a closed rectangular box $R$ is integrable.*

*Proof.* Let $S$ be any closed rectangular box contained in $R$. Define the **span of $f$ on $S$** to be

$$\mathrm{span}_f(S) = \sup(f(S)) - \inf(f(S)).$$

A basic result about the span of continuous functions is given by the following:

**The Small Span Theorem.** *For every $\varepsilon > 0$, there exists a partition $P = \{S_j\}_{j=1}^r$ of $R$ such that $\mathrm{span}_f(S_j) < \varepsilon$, for each $j \leq r$.*

Let us first see how this implies the integrability of $f$ over $R$. Recall that, by a Lemma of section 4.1, it suffices to show that, given any $\varepsilon > 0$, there is a partition $P$ of $R$ such that $U(f,P) - L(f,P) < \varepsilon$. Now by the small span theorem, we can find a partition $P = \{S_j\}$ such that $\mathrm{span}_f(S_j) < \varepsilon'$, for all $j$, where $\varepsilon' = \varepsilon/\operatorname{vol}(R)$. Then clearly,

$$U(f,P) - L(f,P) < \varepsilon' \operatorname{vol}(R) = \varepsilon.$$

Done. □

It now remains to supply a **proof of the small span theorem**. We will prove this by contradiction. Suppose the theorem is false. Then there exists $\varepsilon_0 > 0$ such that, for **every** partition $P = \{S_j\}$ of $R$, $\mathrm{span}_f(S_j) \geq \varepsilon_0$ for some $j$. For simplicity of exposition, we will only treat the case of a rectangle $R = [a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}^2$. The general case is very similar, and can be easily carried out along the same lines with a bit of book-keeping. Divide $R$ into four rectangles by subdividing along the bisectors of $[a_1, b_1]$ and $[a_2, b_2]$. Then for one of these four rectangles, call it $R_1$, we must have that for every partition $\{S_j\}$ of $R_1$ there is a $j$ so that $\mathrm{span}_f(S_j) \geq \varepsilon_0$. Do this again and again, and we finally end up with an infinite sequence of **nested** closed rectangles $R = R_0, R_1, R_2, \ldots$, such that, for every $m \geq 0$, the span of $f$ is at least $\varepsilon_0$ for **any** partition of $P_m = \{S_{j,m}\}$ of $R_m$ on some $S_{j,m}$. Let $z_m = (x_m, y_m)$ denote the southwestern corner of $R_m$, for each $m \geq 0$. Then the sequence $\{z_m\}_{m \geq 0}$ is bounded, and so we may find the least upper bound (**sup**) $\alpha$ (resp. $\beta$) of $x_m$ (resp. $y_m$). Put $\gamma = [\alpha, \beta]$. Then $\gamma \in R$ as the northeastern corner of $R$ is clearly an upper bound of the $z_m$. Since $f$ is continuous at $\gamma$, we can find a non-empty closed rectangular subbox $S$ of $R$ containing $\gamma$ such that $\mathrm{span}_f(S) < \varepsilon_0$. But by construction $R_m$ will have to lie inside $S$ if $m$ is large enough, say for $m \geq m_0$. This gives a contradiction to the span of $f$ being $\geq \varepsilon_0$ on some open set of every partition of $R_{m_0}$. Thus the small span theorem holds for $(f, R)$.

## §4.4 Bounded functions with negligible discontinuities

One is very often interested in being able to integrate bounded functions over $R$ which are continuous except on a very "small" subset. To be precise, we say that a subset $Y$ of $\mathbb{R}^n$ has **content zero** if, for every $\varepsilon > 0$, we can find closed rectangular boxes $Q_1, \ldots, Q_m$ such that

(i) $Y \subseteq \cup_{i=1}^m Q_i$, and
(ii) $\sum_{i=1}^m \operatorname{vol}(Q_i) < \varepsilon$.

**Examples.** (1) A finite set of points in $\mathbb{R}^n$ has content zero. (Proof is obvious!)

(2) Any subset $Y$ of $\mathbb{R}$ which contains a non-empty open interval $(a, b)$ does **not** have content zero.

*Proof.* It suffices to prove that $(a, b)$ has non-zero content for $a < b$ in $\mathbb{R}$. Suppose $(a, b)$ is covered by a finite union of closed intervals $I_i$, $1 \le i \le m$ in $\mathbb{R}$. Then clearly, $S := \sum_{i=1}^m \text{length}(I_i) \ge \text{length}(a, b) = b - a$. So we can never make $S$ less than $b - a$.

(3) The line segment $L = \{x, 0) \mid a_1 < x < b_1\}$ in $\mathbb{R}^2$ has content zero. (Comparing with (2), we see that the notion of content is very much dependent on what the ambient space is, and not just on the set.)

*Proof.* For any $\varepsilon > 0$, cover $L$ by the single closed rectangle

$$R = \left\{ (x, y) \,\middle|\, a_1 \le x \le b_1, \, -\frac{\varepsilon}{4(b_1 - a_1)} \le y \le \frac{\varepsilon}{4(b_1 - a_1)} \right\}.$$

Then $\text{vol}(R) = (b_1 - a_1)\frac{\varepsilon}{2(b_1 - a_1)} = \frac{\varepsilon}{2} < \varepsilon$, and we are done.

The third example leads one to ask if any bounded curve in the plane has content zero. The best result we can prove here is the following

**Proposition.** *Let $\varphi : [a, b] \to \mathbb{R}$ be a* **continuous** *function. Then the graph $\Gamma$ of $\varphi$ has content zero.*

*Proof.* Note that $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b, y = \varphi(x)\}$. Let $\varepsilon > 0$. By the small span theorem, we can find a partition $a = t_0 < t_1 < \cdots < t_r = b$ of $[a, b]$ such that $\text{span}_\varphi([t_{i-1}, t_i]) < \frac{\varepsilon}{(b-a)}$, for every $i = 1, \ldots, r$. Thus the piece of $\Gamma$ lying between $x = t_{i-1}$ and $x = t_i$ can be enclosed in a closed rectangle $S_i$ of area less than $\frac{\varepsilon(t_i - t_{i-1})}{(b-a)}$.

Now consider the collection $\{S_i\}_{1 \le i \le r}$ which covers $\Gamma$. Then we have

$$\sum_{j=1}^r \text{area}(S_j) < \frac{\varepsilon}{(b-a)} \sum_{i=1}^r (t_i - t_{i-1}) = \varepsilon. \quad \square$$

**Theorem.** *Let $f$ be a bounded function on $R$ which is continuous except on a subset $Y$ of content zero. Then $f$ is integrable on $R$.*

*Proof.* Let $M > 0$ be such that $|f(x)| \le M$, for all $x \in R$. Since $Y$ has content zero, we can find closed subrectangular boxes $S_1, \ldots, S_m$ of $R$ such that

(i) $Y \subseteq \cup_{i=1}^m S_i$, and
(ii) $\sum_{i=1}^m \text{vol}(S_i) < \frac{\varepsilon}{4M}$.

Extend $\{S_1, \ldots, S_m\}$ to a partition $P = \{S_1, \ldots, S_r\}$, $m < r$, of $R$. Applying the small span theorem, we may suppose that $S_{m+1} \ldots, S_r$ are so chosen that (for each $i \geq m+1$) $\mathrm{span}_f(S_i) < \frac{\varepsilon}{2\,\mathrm{vol}(R)}$. (We can apply this theorem because $f$ is continuous outside the union of $S_1, \ldots, S_m$.) So we have

$$U(f,P) - L(f,P) \leq 2M \sum_{i=1}^{m} \mathrm{vol}(S_i) + \sum_{i=m+1}^{r} \mathrm{span}_f(S_i)\,\mathrm{vol}(S_i)$$
$$< (2M)\left(\frac{\varepsilon}{2M}\right) + \frac{\varepsilon}{2\,\mathrm{vol}(R)} \sum_{i=m+1}^{r} \mathrm{vol}(S_i).$$

But the right hand side is $\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, because $\sum_{i=m+1}^{r} \mathrm{vol}(S_i) \leq \mathrm{vol}(R)$. $\square$

**Example.** Let $R = [0,1] \times [0,1]$ be the unit square in $\mathbb{R}^2$, and $f : R \to \mathbb{R}$ the function defined by $f(x,y) = x + y$ if $x \leq y$ and $x - y$ if $x \geq y$. Show that $f$ is integrable on $R$.

Let $D = \{(x,x) \mid 0 \leq x \leq 1\}$ be the "diagonal" in $R$. Then $D$ has content zero as it is the graph of the continuous function $\varphi(x) = x$, $0 \leq x \leq 1$. Moreover, $f$ is discontinuous only on $D$. So $f$ is continuous on $R - D$ with $D$ of content zero, and consequently $f$ is integrable on $R$.

*Remark.* We can use this theorem to define the integral of a **continuous function $f$ on any closed bounded set** $B$ in $\mathbb{R}^n$ **if the boundary** of $B$ **has content zero**. Indeed, in such a case, we may enclose $B$ in a closed rectangular box $R$ and define a function $\tilde{f}$ on $R$ by making it equal $f$ on $B$ and 0 on $R - B$. Then $\tilde{f}$ will be continuous on all of $R$ except for the boundary of $B$, which has content zero. So $\tilde{f}$ is integrable on $R$. Since $\tilde{f}$ is 0 outside $B$, it is reasonable to set

$$\int_B f = \int_R \tilde{f}.$$

It is often useful to consider a finer notion than content, called *measure*. Before giving this definition recall that a set $X$ is countable iff there is a bijection (or as some would say, *one-to-one correspondence*, between $X$ and a subset of the set $\mathbb{N}$ of natural numbers. Check that $\mathbb{Z}$ and $\mathbb{Q}$ are countable, while $\mathbb{R}$ is not.

A subset $Y$ of $\mathbb{R}^n$ is said to have **measure zero** if, for every $\varepsilon > 0$, we can find a countable collection of closed rectangular boxes $Q_1, Q_2, \ldots, Q_m, \ldots$ such that

(i) $Y \subseteq \cup_{i \geq 1} Q_i$, and
(ii) $\sum_{i \geq 1} \mathrm{vol}(Q_i) < \varepsilon$.

One can use open rectangular boxes instead of closed ones, and the resulting definition will be equivalent.

**Examples.** (1) A countable set of points in $\mathbb{R}^n$ has measure zero.

(2) Any subset $Y$ of $\mathbb{R}$ which contains a non-empty open interval $(a,b)$ does **not** have measure zero.

(3) A countable union of lines in $[0,1] \times [0,1] \subset \mathbb{R}^2$ has measure zero.

We will state the following result without proof:

**Theorem.** *Let $R$ be a closed rectangular box in $\mathbb{R}^n$, and $f$ a bounded function on $R$ which is continuous except on a subset $Y$ of measure zero. Then $f$ is integrable on $R$.*

## §4.5 Fubini's theorem

So far we have been meticulous in figuring out when a given bounded function $f$ is integrable on $R$. But if $f$ is integrable, we have developed no method whatsoever to actually find a way to integrate it except in the really easy case of a step function. We propose to ameliorate the situation now by describing a very reasonable and computationally helpful result. We will state it in the plane, but there is a natural analog in higher dimensions as well. In any case, many of the intricacies of multiple integration are present already for $n = 2$, and it is a wise idea to understand this case completely at first.

**Theorem (Fubini).** *Let $f$ be a bounded, integrable function on $R = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$. For $x$ in $[a_1, b_1]$, put $A(x) = \int_{a_2}^{b_2} f(x,y)\, dy$ and assume the following*

(i) *$A(x)$ exists for each $x \in [a_1, b_1]$, i.e., the function $y \mapsto f(x,y)$ is integrable on $[a_2, b_2]$ for any fixed $x$ in $[a_1, b_1]$;*
(ii) *$A(x)$ is integrable on $[a_1, b_1]$.*

*Then*

$$\iint_R f(x,y)\, dxdy = \int_{a_1}^{b_1} \left( \int_{a_2}^{b_2} f(x,y)\, dy \right) dx.$$

In other words, once the hypotheses (i) and (ii) are satisfied, we can compute $\int_R f$ by performing two 1-dimensional integrals in order. One cannot always reverse the order of integration, however, and if one wants to integrate over $x$ first, one needs to assume the obvious analog of the conditions (i), (ii).

*Proof.* Let $P_1 = \{B_i \mid 1 \leq i \leq \ell\}$ (resp. $P_2 = \{C_j \mid 1 \leq j \leq m\}$) be a partition of $[a_1, b_1]$ (resp. $[a_2, b_2]$), with $B_i, C_j$ closed intervals in $\mathbb{R}$. Then $P = P_1 \times P_2 = \{B_i \times C_j\}$ is a partition of $R$. By hypothesis (i), we have

$$L(f_x, P_2) \leq A(x) \leq U(f_x, P_2),$$

where $f_x$ is the one-dimensional function $y \mapsto f(x,y)$. Then applying hypothesis (ii), we get

$$L(L(f_x, P_2), P_1) \leq \int_{a_1}^{b_1} A(x)\, dx \leq U(U(f_x, P_2), P_1).$$

But we have

$$\begin{aligned} L(L(f_x, P_2), P_1) &= \sum_{i=1}^{\ell} \mathrm{length}(B_i) \inf(L(f_x, P_2)(B_i)) \\ &= \sum_{i=1}^{\ell} \mathrm{length}(B_i) \sum_{j=1}^{m} \mathrm{length}(C_j) \inf(f(B_i \times C_j)) = L(f, P). \end{aligned}$$

Similarly for the upper sum. Hence $L(f, P) \le \int_{a_1}^{b_1} A(x)\,dx \le U(f, P)$. Given any partition $Q$ of $R$, we can find a partition $P$ of the form $P_1 \times P_2$ which refines $Q$. Thus $L(f, Q) \le \int_{a_1}^{b_1} A(x)\,dx \le U(f, Q)$, for **every** partition $Q$ of $R$. Then by the uniqueness of $\int_R f$, which exists because $f$ is integrable, $\int_{a_1}^{b_1} A(x)\,dx$ is forced to be $\int_R f$. $\square$

*Remark.* The reason we denote $\int_{a_2}^{b_2} f(x, y)\,dy$ by $A(x)$ is the following. The double integral $\iint_R f(x, y)\,dxdy$ is the **volume** subtended by the graph $\Gamma = \{(x, y, f(x, y)) \in \mathbb{R}^3\}$ over the rectangle $R$. (Note that $\Gamma$ is a "surface" since $f$ is a function of two variables.) When we fix $x$ at the same point $x_0$ in $[a_1, b_1]$, the intersection of the plane $\{x = x_0\}$ with $\Gamma$ in $\mathbb{R}^3$ is a curve, which is none other than the graph $\Gamma_{x_0}$ of $f_{x_0}$ in the $(y, z)$-plane shifted to $x = x_0$. The **area** under $\Gamma_{x_0}$ over the interval $[a_2, b_2]$ is just $\int_{a_2}^{b_2} f_{x_0}(y)\,dy$; whence the name $A(x_0)$. Note also that as $x_0$ goes from $a_1$ to $b_1$, the whole volume is swept by the slice of area $A(x_0)$.

A natural question to ask at this point is whether the hypotheses (i), (ii) of Fubini's theorem are satisfied by many functions. The answer is yes, and the prime examples are continuous functions.

**Theorem.** *Let $f$ be a continuous function on $R = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$. Then $\int_R f$ can be computed as an iterated integral in either order. To be precise, we have*

$$\iint_R f(x, y)\,dxdy = \int_{a_1}^{b_1} \left[ \int_{a_2}^{b_2} f(x, y)\,dy \right] dx = \int_{a_2}^{b_2} \left[ \int_{a_1}^{b_1} f(x, y)\,dx \right] dy.$$

*Proof.* Since $f$ is continuous on (the compact set) $R$, it is certainly bounded. Let $M > 0$ be such that $|f(x, y)| \le M$. We have also seen that it is integrable. For each $x$, the function $y \mapsto f(x, y)$ is integrable on $[a_2, b_2]$ because of continuity on $[a_2, b_2]$. So we get hypothesis (i) of Fubini. To get hypothesis (ii), it suffices to show that $A(x) = \int_{a_2}^{b_2} f(x, y)\,dy$ is continuous in $x$. For $h$ small, we have

$$|A(x+h) - A(x)| = \left| \int_{a_2}^{b_2} (f(x+h, y) - f(x, y))\,dy \right| \le \int_{a_2}^{b_2} |f(x+h, y) - f(x, y)|\,dy.$$

By the small span theorem we can find a partition $\{S_j\}$ of $R$ with $\mathrm{span}_f(S_j) < \varepsilon/(b_2 - a_2)$. If $h$ is small enough so that $(x + h, y)$ and $(x, y)$ lie in the same box for all $y$

(which we can achieve since $x$ is fixed and there are only finitely many boxes) we have $|f(x+h,y)-f(x,y)| < \mathrm{span}_f(S_j) < \varepsilon/(b_2-a_2)$ where $S_j$ is a box containing both points. Note that this argument also works if $(x,y)$ lies on the vertical boundary between two boxes: for positive $h$ we land in one box and for negative $h$ in the other. Hence

$$\int_{a_2}^{b_2} |f(x+h,y)-f(x,y)|\, dy < \varepsilon$$

for $h$ sufficiently small. This shows that $A(x)$ is continuous and hence integrable on $[a_1,b_1]$. We have now verified both hypotheses of Fubini, and hence

$$\iint_R f(x,y)\, dxdy = \int_{a_1}^{b_1} \left[\int_{a_2}^{b_2} f(x,y)\, dy\right] dx.$$

To prove that $\int_R f$ is also computable using the iteration in reverse order, all we have to do is note that by a symmetrical argument, the integral $\int_{a_1}^{b_1} f(x,y)\, dx$ makes sense and is continuous in $y$, hence integrable on $[a_2,b_2]$. The Fubini argument then goes through. $\square$

*Remark.* We will note the following extension of the theorem above without proof.

Let $f$ be a continuous function on a closed rectangular box $R = [a_1,b_1]\times\cdots\times[a_n,b_n]$. Then the integral of $f$ over $R$ is computable as an iterated integral

$$\int_{a_1}^{b_1}\left[\cdots\left[\int_{a_{n-1}}^{b_{n-1}}\left[\int_{a_n}^{b_n} f(x_1,\ldots,x_n)\, dx_n\right] dx_{n-1}\right]\ldots\right] dx_1.$$

Moreover, we can compute this in any order we want, e.g., integrate over $x_2$ first, then over $x_5$, then over $x_1$, etc. Note that there are $n!$ possible ways here of permuting the order of integration.

## §4.6 Integration over special regions

Let $Z$ be a **compact set in** $\mathbb{R}^n$. Since it is bounded, we may enclose it in a closed rectangular box $R$. If $f$ is a bounded function on $Z$, we may define an extension $\tilde{f}$ to $R$ by setting $\tilde{f}(x)$ to be $f(x)$ (resp. 0) for $x$ in $Z$ (resp. in $R-Z$).

Let us say that $f$ **is integrable over** $Z$ if $\tilde{f}$ is integrable over $R$, and put

$$\int_Z f = \int_R \tilde{f}.$$

It is clear that this definition is independent of the choice of $R$. In §4.4, where we introduced the notion of content, we remarked that if the boundary of $Z$ had content zero and if $f$ is continuous, then $\tilde{f}$ would be integrable on $R$. The same idea easily proves the following

**Theorem.** *Let $Z$ be a compact subset of $\mathbb{R}^n$ such that the* **boundary of $Z$ has content zero**. *Then* **any** *function $f$ on $Z$ which is* **continuous** *on $Z$ is integrable over $Z$.*

In fact, one can replace content by measure in this Theorem.

Now we will analyze the simplest cases of this phenomenon in $\mathbb{R}^2$.

**Definition.** A **region of type I** in $\mathbb{R}^2$ is a set of the form $\{a \leq x \leq b,\, \varphi_1(x) \leq y \leq \varphi_2(x)\}$, where $\varphi_1, \varphi_2$ are continuous functions on $[a, b]$.

A **region of type II** in $\mathbb{R}^2$ is a set of the form $\{c \leq y \leq d,\, \psi_1(y) \leq x \leq \psi_2(y)\}$, where $\psi_1, \psi_2$ are continuous functions on $[c, d]$.

A **region of type III** in $\mathbb{R}^2$ is a subset which is simultaneously of type I and type II.

*Remark.* Note that a circular region is of type III.

**Theorem.** *Let $f$ be a continuous function on a subset $S$ of $\mathbb{R}^2$.*

(a) *Suppose $S$ is a region of type I defined by $a \leq x \leq b$, $\varphi_1(x) \leq y \leq \varphi_2(x)$, with $\varphi_1, \varphi_2$ continuous. Then $f$ is integrable on $S$ and*

$$\int_S f = \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \right) dx.$$

(b) *Suppose $S$ is a region of type II defined by $c \leq y \leq d$, $\psi_1(y) \leq x \leq \psi_2(y)$, with $\psi_1, \psi_2$ continuous. Then $f$ is integrable on $S$ and*

$$\int_S f = \int_c^d \left( \int_{\psi_1(x)}^{\psi_2(y)} f(x, y)\, dx \right) dy.$$

*Proof.* We will prove (a) and leave the symmetrical case (b) to the reader.

(a) Let $R = [a, b] \times [c, d]$, where $c, d$ are chosen so that $R$ contains $S$. Define $\tilde{f}$ on $R$ as above (by extension of $f$ by zero outside $S$). By the Proposition of §4.4, we know that the graphs of $\varphi_1$ and $\varphi_2$ are of content zero, since $\varphi_1, \varphi_2$ are continuous. Thus the main theorem of §4.4 implies that $\tilde{f}$ is integrable on $R$ as its set of discontinuities is contained in the boundary $\partial S$ of $S$. It remains to prove that $\int_S f$ $(= \int_R \tilde{f})$ is given by the iterated integral $\int_a^b (\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy)\, dx$. For each $x \in (a, b)$, the integral $\int_c^d \tilde{f}(x, y)\, dy$ exists as the set of discontinuities in $[c, d]$ has at most two points. Moreover, the function $x \mapsto \int_c^d \tilde{f}(x, y),\, dy$ is integrable on $[a, b]$. Hence (the proof of) Fubini's theorem applies in this context and gives

$$\int_R f = \int_a^b \left( \int_c^d \tilde{f}(x, y)\, dy \right) dx.$$

Since the inside integral (over $y$) is none other than $\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy$, the assertion of the theorem follows. $\square$

## §4.7 Examples

(1) Compute $\int_R f$, where $R$ is the closed rectangle $[-1,1]\times[2,3]$ and $f$ the function $(x,y)\mapsto x^2y - x\cos\pi y$. Since $f$ is continuous on $R$, we may apply Fubini's theorem and compute $\int_R I$ as the iterated integral

$$I = \int_{-1}^{1}\left(\int_2^3 (x^2y - x\cos\pi y)\,dy\right)dx.$$

Recall that in $\int_2^3 (x^2y - x\cos\pi y)\,dy$, $x$ is treated like a constant, hence equals

$$x^2\int_2^3 y\,dy - x\int_2^3 \cos\pi y\,dy = x^2\left(\frac{3^2}{2}-\frac{2^2}{2}\right) - x\left(\frac{\sin\pi y}{\pi}\right)\Bigg]_2^3 = \frac{5}{2}x^2.$$

$$\Rightarrow I = \frac{5}{2}\int_{-1}^{1} x^2\,dx = \frac{5}{2}\left(\frac{x^3}{3}\right)\Bigg]_{-1}^{1} = \frac{5}{3}\,.$$

We could also have computed it in the opposite order to get

$$\begin{aligned} I &= \int_2^3\left[\int_{-1}^{1}(x^2y - x\cos\pi y)\,dx\right]dy \\ &= \int_2^3\left(y\left(\frac{x^3}{3}\right)\Bigg]_{-1}^{1} - \cos\pi y\left(\frac{x^2}{2}\right)\Bigg]_{-1}^{1}\right)dy \\ &= \int_2^3\left(\frac{2y}{3}\right)dy = \frac{y^2}{3}\Bigg]_2^3 = \frac{5}{3}\,. \end{aligned}$$

(2) Find the volume of the tetrahedron $T$ in $\mathbb{R}^3$ bounded by the planes $x=0$, $y=0$, $z=0$ and $x-y-z=-1$.

Note first that the base of $T$ is a triangle $\triangle$ defined by $-1\le x\le 0$, $0\le y\le x+1$. Given any $(x,y)$ in $\triangle$, the height of $T$ above it is simply given by $z = x-y+1$. Hence we get by the Theorem of §4.6,

$$\begin{aligned} \mathrm{vol}(T) &= \iint_\triangle (x-y+1)\,dxdy = \int_{-1}^{0}\left(\int_0^{x+1}(x-y+1)\,dy\right)dx \\ &= \int_{-1}^{0}\left(xy - \frac{y^2}{2} + y\right)\Bigg]_0^{x+1} dx \\ &= \int_{-1}^{0}\frac{(x+1)^2}{x}\,dx = \int_0^1 \frac{u^2}{2}\,du = \frac{1}{6}\,. \end{aligned}$$

(3) Fix $a,b>0$, and consider the region $S$ inside the ellipse defined by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in $\mathbb{R}^2$. Compute $I = \iint_S \sqrt{a^2-x^2}\,dxdy$.

Note that $S$ is a region of type I as we may write it as

$$\left\{-a \le x \le a,\ -b\sqrt{1-\frac{x^2}{a^2}} \le y \le b\sqrt{1-\frac{x^2}{a^2}}\right\}.$$

Since the function $(x, y) \mapsto \sqrt{a^2 - x^2}$ is continuous, we can apply the main theorem of §4.6. We obtain

$$\begin{aligned} I &= \int_{-a}^{a} \sqrt{a^2 - x^2} \left( \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b-\sqrt{1-\frac{x^2}{a^2}}} dy \right) dx \\ &= \frac{2b}{a} \int_{-a}^{a} (a^2 - x^2)\, dx = \frac{2b}{a} \left( a^2 x - \frac{x^3}{3} \right) \Big]_{-a}^{a} \\ &= 4a^2 b - \frac{4a^2 b}{3} = \frac{8a^2 b}{3}. \end{aligned}$$

## §4.8 Applications

Let $S$ be a thin plate in $\mathbb{R}^2$ with matter distributed with density $f(x, y)$ (= mass/unit area). The **mass of $S$** is given by

$$m(S) = \iint_S f(x, y)\, dx dy.$$

The **average density** is

$$\frac{m(S)}{\text{area}} = \frac{\iint_S f(x, y)\, dx dy}{\iint_S dx dy}.$$

The **center of mass** of $S$ is given by $\bar{z} = (\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{m(S)} \iint_S x\, f(x, y)\, dx dy$$

and

$$\bar{y} = \frac{1}{m(S)} \iint_S y\, f(x, y)\, dx dy.$$

When the **density is constant**, the center of mass is called the **centroid** of $S$.

Suppose $L$ is a fixed line. For any point $(x, y)$ on $S$, let $\delta = \delta(x, y)$ denote the (perpendicular) distance from $(x, y)$ to $L$. The **moment of inertia about $L$** is given by

$$I_L = \iint_S \delta^2(x, y)\, f(x, y)\, dx dy.$$

When $L$ is the $x$-axis (resp. $y$-axis), it is customary to write $I_x$ (resp. $I_y$).

Note that the center of mass is a linear invariant, while the moment of inertia is quadratic.

An interesting use of the centroid occurs in the computation of volumes of revolutions. To be precise we have the following

**Theorem (Pappus).** *Let $S$ be a region of type I, i.e., given as $\{a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}$, with $\varphi_1, \varphi_2$ continuous. Suppose that $\min_x \varphi_1(x) > 0$, so that $S$ lies above the $x$-axis. Denote by $V$ the volume of the solid $M$ obtained by revolving $S$ about the $x$-axis, and by $\bar{z} = (\bar{x}, \bar{y})$ the centroid of $S$. Then*

$$V = 2\pi \bar{y}\, a(S),$$

*where $a(S)$ is the area of $S$.*

*Proof.* Let $V_i$ denote the volume of the solid obtained by revolving $\{(x, \varphi_1(x) \mid a \leq x \leq b\}$ about the $x$-axis. Then

$$V_i = \pi \int_a^b \varphi_i(x)^2\, dx.$$

(This is a result from one-variable calculus.) But clearly, $V = V_2 - V_1$. So we have

$$V = \pi \int_a^b [\varphi_2(x)^2 - \varphi_1(x)^2]\, dx.$$

On the other hand, we have by the definition of the centroid,

$$\bar{y} = \frac{1}{a(S)} \iint_S y\, dxdy.$$

Since $y$ is continuous and $S$ a region of type I, we have

$$\begin{aligned} \bar{y} &= \frac{1}{a(S)} \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} y\, dy \right) dx \\ &= \frac{1}{a(S)} \int_a^b \tfrac{1}{2} [\varphi_2(x)^2 - \varphi_1(x)^2]\, dx. \end{aligned}$$

The theorem now follows immediately. □

**Examples.** (1) Let $S$ be the semi-circular region $\{-1 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\}$. Compute the centroid of $S$.

Since $S$ is of type I, we have

$$\begin{aligned} a(S) &= \iint_S dxdy = \int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} dy \\ &= \int_{-1}^{1} \sqrt{1-x^2}\, dx = 2 \int_0^1 \sqrt{1-x^2}\, dx. \end{aligned}$$

Put $x = \sin t$, $0 \leq t \leq \frac{\pi}{2}$. Then $dx = \cos t\, dt$ and $\sqrt{1-x^2} = \cos t$. So we get

$$\begin{aligned} a(S) &= 2 \int_0^{\frac{\pi}{2}} \cos^2 t\, dt = 2 \int_0^{\frac{\pi}{2}} \left( \frac{1+\cos t}{2} \right) dt \\ &= 2 \left[ \frac{\pi}{4} + \frac{\sin 2t}{4} \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}\,. \end{aligned}$$

Of course, we could have directly reasoned by geometry that the area of a semi-circular region of radius 1 is $\frac{\pi}{2}$.

Let $\bar{z} = (\bar{x}, \bar{y})$ be the centroid.

$$\begin{aligned}\bar{x} &= \frac{1}{a(S)} \iint_S x\, dxdy = \frac{2}{\pi} \int_{-1}^{1} \left( \int_0^{\sqrt{1-x^2}} dy \right) x\, dx \\ &= \frac{2}{\pi} \int_{-1}^{1} x\sqrt{1-x^2}\, dx = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t \cos^2 t\, dt = 0,\end{aligned}$$

since the integrand is an odd function.

Again, the fact that $\bar{x} = 0$ can be directly seen by geometry. The key thing is to compute $\bar{y}$. We have

$$\begin{aligned}\bar{y} &= \frac{2}{\pi} \int_{-1}^{1} dx \left( \int_0^{\sqrt{1-x^2}} y\, dy \right) = \frac{1}{\pi} \int_{-1}^{1} (1 - x^2)\, dx \\ &= \frac{1}{\pi} \left( x - \frac{x^3}{3} \right) \bigg]_{-1}^{1} = \frac{2}{\pi} - \frac{2}{3\pi} = \frac{4}{3\pi}\,.\end{aligned}$$

So the centroid of $S$ is $(0, \frac{4}{3\pi})$.

(2) Find the volume $V$ of the **torus** $\pi$ obtained by revolving about the $x$-axis a circular region $S$ of radius $r$ (lying above the $x$-axis).

The area $a(S)$ is $\pi r^2$, and the centroid $(\bar{x}, \bar{y})$ is located at the center of $S$ (easy check!). Let $b$ be the distance from the center of $S$ to the $x$-axis. Then by Pappus' theorem,

$$V = 2\pi\bar{y}\, a(S) = 2\pi b(\pi r^2) = 2\pi^2 r^2 b.$$

# Chapter 5

# Line Integrals

A basic problem in higher dimensions is the following. Suppose we are given a function (scalar field) $g$ on $\mathbb{R}^n$ and a bounded curve $C$ in $\mathbb{R}^n$, can one define a suitable integral of $g$ over $C$? We can try the following at first. Since $C$ is bounded, we can enclose it in a closed rectangular box $R$ and then define a function $\tilde{g}$ on $R$ to be $g$ on $C$ and 0 outside. Clearly, $\int_R \tilde{g} = \int_C g$, but for $n > 1$, this integral will usually give only zero because $C$ will in general have measure zero. (We say in general, because there are space-filling curves.) So we have to do something different. Before giving the "right" definition, let us review the basic properties of a curve $C$ in $\mathbb{R}^n$, parametrized by an interval in $\mathbb{R}$.

Recall that a (parametrized) curve $C$ in $\mathbb{R}^n$ is the image of a continuous map $\alpha : [a, b] \to \mathbb{R}^n$, for some $a < b$ in $\mathbb{R}$. Such a curve is said to be $\mathcal{C}^1$ if and only if $\alpha'(t)$ exists and is continuous everywhere on the interval. We like this condition because it implies that the curve $C$ has a well defined unit tangent vector $T$ at every point where $\alpha'(t) \neq 0$, and moreover, that this $T$ moves continuously as we move along the curve. (If $\alpha'(t) \neq 0$, we take $T$ to be $\alpha'(t)/||\alpha'(t)||$.) In practice, we will be able to allow a finite number of points where the curve develops corners. This leads to the definition of a **piecewise** $\mathcal{C}^1$ curve $C$ to be a curve which is a **finite** union of $\mathcal{C}^1$ curves $C_1, C_2, \ldots, C_r$.

The **arc length** of a $\mathcal{C}^1$ curve $C$ parametrized by $\alpha : [a, b] \to \mathbb{R}^n$ is defined at any point $t \in [a, b]$ to be

$$s(t) = \int_a^t \|\alpha'(t)\| \, dt.$$

$s(b)$ is called the arc length of the whole curve.

We will now describe the heuristic reason for this definition.

Let $Q$ be a partition of $[a, b]$, i.e., the datum $a = t_0 < t_1 < \cdots < t_m = b$. Let $L_i$ denote

the line joining $\alpha(t_{i-1})$ to $\alpha(t_i)$ in $\mathbb{R}^n$, $\forall\, i \geq 1$. Put

$$\ell_m = \sum_{i=1}^{m} \text{length}(L_i),$$

which equals $\sum_{i=1}^{m} \|\alpha(t_i) - \alpha(t_{i-1})\|$.

Clearly, as $m$ is very large, $\ell_m$ provides a good approximation for $s(b)$, and a proper definition must surely express $s(b)$ as the limit of $\ell_m$ as $m$ tends to infinity.

For each $i$, we may apply the mean value theorem to $\alpha(t)$ restricted to $[t_{i-1}, t_i]$ and find a point $z_i$ in that subinterval such that

$$\alpha(t_i) - \alpha(t_{i-1}) = (t_i - t_{i-1})\alpha'(z_i).$$

This yields the equality

$$\ell_m = \sum_{i=1}^{m} (t_i - t_{i-1}) \|\alpha'(z_i)\|.$$

Since by assumption $\alpha'$ is continuous, we may let $m$ go to infinity and get

$$\lim_{m \to \infty} \ell_m = \int_a^b \|\alpha'(t)\| \, dt.$$

**Examples:** (1) $C$ = circle in $\mathbb{R}^2$ with radius $r$ and center 0. It is parametrized by $\alpha(t) = (r\cos t, r\sin t)$, $0 \leq t \leq 2\pi$. Since $\sin t$ and $\cos t$ are continuously differentiable, $C$ is $\mathcal{C}^1$. Moreover, $\alpha'(t) = (-r\sin t, r\cos t)$, and so $\|\alpha'(t)\| = r\sqrt{(-\sin t)^2 + (\cos t)^2} = r$. Hence the arc length of $C$ is $\int_0^{2\pi} r\, dt = 2\pi r$, as expected. (It is always good to do such a simple example at first, because it tells us that the theory is seemingly on the right track.)

(2) (Helix) Let $C$ in $\mathbb{R}^3$ be parametrized by $\alpha(t) = (\cos t, \sin t, t)$, $0 \leq t \leq 4\pi$. Again, $\alpha$ is (certainly) $\mathcal{C}^1$, and $\alpha'(t) = (-\sin t, \cos t, 1)$. So $\|\alpha'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$, and the arc length of $C$ is $\int_0^{4\pi} \sqrt{2}\, dt = 4\pi\sqrt{2}$.

(3) Let $C$ be defined by $\alpha(t) = (2t, t^2, \log t) \in \mathbb{R}^3$ for $t \in [\frac{1}{4}, 2000]$. Suppose we need to find the arc length $\ell$ between the points $(2, 1, 0)$ and $(4, 4, \log 2)$. Note first that $(2, 1, 0) = \alpha(1)$ and $(4, 4, \log 2) = \alpha(2)$. So we have:

$$\ell = \int_1^2 \|\alpha'(t)\| \, dt.$$

Also, $\alpha'(t) = (2, 2t, \frac{1}{t})$, so $\|\alpha'(t)\| = \sqrt{4 + 4t^2 + \frac{1}{t^2}} = (2t + \frac{1}{t})$. (Note that since $t > 0$, we do not need to take the absolute value while taking the square root.) Hence

$$\ell = \int_1^2 2t\,dt + \int_1^2 \frac{1}{t}\,dt = t^2\Big]_1^2 + \log t\Big]_1^2 = 3 + \log 2.$$

(4) This example deals with a piecewise $\mathcal{C}^1$ curve. Note that the definition of arc length goes over in the obvious way for such curves.

Take $C$ to be the unit (upper) semicircle $C_1$ centered at 0, together with the flat diameter $C_2$, then we can parametrize $C_1$ in the usual way by setting $\alpha_1(t) = (\cos t, \sin t)$, for $t$ in $[0, \pi]$. Let us parametrize $C_2$ by a function $\alpha_2$ on $[\pi, 2\pi]$ as follows. Write $\alpha_2(t) = (dt + e, 0)$ with the requirement $\alpha_2(\pi) = (-1, 0)$ and $\alpha_2(2\pi) = (1, 0)$. Then we need $d\pi + e = -1$ and $2d\pi + e = 1$, which yields $d = \frac{2}{\pi}$ and $e = -3$. So we have: $\alpha_2(t) = (\frac{2}{\pi}t - 3, 0)$ for $t \in [\pi, 2\pi]$. Clearly, $\alpha_1$ and $\alpha_2$ are both $\mathcal{C}^1$, so $C$ is piecewise $\mathcal{C}^1$.

Suppose we want to find the arc length $\ell(C)$ of the whole curve $C$. It is clear from geometry that $\ell(C_1) = \pi$ and $\ell(C_2) = 2$, so $\ell(C) = \pi + 2$. We can also derive this from Calculus. Indeed, $\|\alpha_1'(t)\| = 1$ and $\|\alpha_2'(t)\| = \|\frac{2}{\pi}\| = \frac{2}{\pi}$. So

$$\ell(C) = \int_0^\pi 1\,dt + \int_\pi^{2\pi} \frac{2}{\pi}\,dt = \pi + 2.$$

Now we are ready to give a way to integrate functions (scalar fields) over curves in $n$-space.

Let $g : \mathcal{D} \to \mathbb{R}$ be a scalar field, with $C \subseteq \mathcal{D} \subseteq \mathbb{R}^n$, where $C$ is a $\mathcal{C}^1$ curve parametrized by $\alpha : [a, b] \to \mathbb{R}^n$. Then we define the **line integral of $g$ over $C$ with respect to arc length** to be

$$\int_C g\,ds = \int_a^b g(\alpha(t))\|\alpha'(t)\|\,dt.$$

Note that by the definition of $s(t)$, $s'(t)$ is none other than $\|\alpha'(t)\|$; so $\|\alpha'(t)\|\,dt$ may be thought of as $ds$.

There is yet another type of line integral. Suppose $f : \mathcal{D} \to \mathbb{R}^n$ is a vector field with $\mathcal{D} \subseteq \mathbb{R}^n$. (Note that the source space and the target space of $f$ have the same dimension.) Then we may define the **line integral of $f$ over $C$** to be

$$\int_C f \cdot d\alpha = \int_a^b f(\alpha(t)) \cdot \alpha'(t)\,dt.$$

These two integrals are related as follows. Let $T$ denote the unit tangent vector to $C$ at $\alpha(t)$. Then $T = \frac{\alpha'(t)}{\|\alpha'(t)\|}$ (assuming $\alpha'(t) \neq 0$). Since $\|\alpha'(t)\| = s'(t)$, we can identify $T$ with $\alpha'(s)$ $(= \frac{d\alpha}{ds})$. Put $g(\alpha(t)) = f(\alpha(t)) \cdot T$, which defines a scalar field. Then we have

$$\int_C f \cdot d\alpha = \int_C g(\alpha(t))\|\alpha'(t)\|\, dt = \int_C g\, ds.$$

The following lemma is immediate from the definition.

**Lemma.** (i) $\int_C f_1 \cdot d\alpha + \int_C f_2 \cdot d\alpha = \int_C (f_1 + f_2) \cdot d\alpha$

(ii) $\int_{C_1} f \cdot d\alpha_1 + \int_{C_2} f \cdot d\alpha_2 = \int_C f \cdot d\alpha$, *where $C$ is the union of $C_1$ and $C_2$, parametrized by $\alpha(t)$ defined by putting together $\alpha_1$ and $\alpha_2$.*

It is important to note that each parametrized curve $C$ comes with an **orientation** (or **direction**). This was not important when we calculated the arc length, but it is quite important to be aware of when line integrals of vector fields are involved.

Suppose $C$ is parametrized in two different ways, say by $\alpha : [a, b] \to \mathbb{R}^n$ and $\beta : [c, d] \to \mathbb{R}^n$. We say that $\alpha$ is equivalent to $\beta$, and write $\alpha \sim \beta$, if there exists a *change of parameter* $\mathcal{C}^1$ function $u : [a, b] \to [c, d]$ such that

(i) $u$ is onto,

(ii) $u'$ is never 0, and

(iii) $\beta(u(t)) = \alpha(t)$, for any $t \in [a, b]$.

It is not hard to see that

$$\int_C f \cdot d\beta = \delta \int_C f \cdot d\alpha,$$

where $\delta$ is the sign of $u'$ on $[a, b]$. For example, $\delta = -1$ when $\alpha(t) = \beta(-t + a + b)$, $a \leq t \leq b$.

## 5.1 An application

Line integrals arise all over the place. We will content ourselves with describing the example of a thin **wire** $W$ in the shape of a $\mathcal{C}^1$ curve $C$ parametrized by $\alpha : [a, b] \to \mathbb{R}^n$. Suppose the **mass density** of $W$ is given by a function $g(x)$, $x = (x_1, \ldots, x_n)$. Then the **total mass** of $W$ is given by

$$M = \int_C g\, ds.$$

Its **center of mass** $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is given by

$$\bar{x}_j = \frac{1}{M} \int_C x_j g \, ds, \qquad 1 \leq j \leq n.$$

When $g$ is a constant field, $W$ is said to be **uniform** and $\bar{x}$ is called the **centroid**.

Suppose $L$ is any fixed line in $\mathbb{R}^n$. Denote by $\delta(x)$ the distance from $x$ to $L$. Then the **moment of inertia about** $L$ is defined to be

$$I_L = \int_C \delta^2 g \, ds.$$

**Example.** Consider the wire $W$ in the shape of $C = C_1 \cup C_2$, with $C_1$ being parametrized by $\alpha_1(t) = (\cos t, \sin t)$, $0 \leq t \leq \pi$, and $C_2$ by $\alpha_2(t) = (t, 0)$, $-1 \leq t \leq 1$. Suppose $W$ has mass density given by $g(x, y) = \sqrt{x^2 + y^2}$. Then $g(\alpha_1(t)) = 1$ and $g(\alpha_2(t)) = |t|$. Also, $\|\alpha_1'(t)\| = 1$ and $\|\alpha_2'(t)\| = 1$. Thus the total mass is given by

$$M = \int_0^\pi 1 \, dt - \int_{-1}^0 t \, dt + \int_0^1 t \, dt = \pi + 1.$$

The coordinates of the center of mass are given by

$$\bar{x} = \frac{1}{\pi + 1} \left( \int_0^\pi \cos t \, dt - \int_{-1}^0 t^2 \, dt + \int_0^1 t^2 \, dt \right) = 0,$$

and

$$\bar{y} = \frac{1}{\pi + 1} \left( \int_0^\pi \sin t \, dt + 0 \right) = \frac{2}{\pi + 1}.$$

The distance from $(x, y)$ to the $x$-axis (resp. $y$-axis) is simply $|y|$ (resp. $|x|$). Hence the **moment of inertia about the $x$-axis** is

$$I_x = \int_0^\pi \sin^2 t \, dt + 0 = \int_0^\pi \left( \frac{1 - \cos 2t}{2} \right) dt = \frac{\pi}{2}.$$

The **moment of inertia about the $y$-axis** is

$$I_y = \int_0^\pi \cos^2 t \, dt - \int_{-1}^0 t^3 \, dt + \int_0^1 t^3 \, dt = \frac{\pi}{2} + \frac{1}{2}.$$

## 5.2 Gradient fields

Given a differentiable scalar field $g$ on an open set $\mathcal{D}$ in $\mathbb{R}^n$, we may of course consider its gradient field $\nabla g$. It turns out, as shown below, that under mild hypotheses on $\nabla g$ and $\mathcal{D}$, the line integrals $\int_C \nabla g \cdot d\alpha$ can be evaluated simply. This result is sometimes called the **second fundamental theorem of Calculus for line integrals**. Before stating and proving the result, we need to introduce an important concept called **connectivity**.

An open set $S$ in $\mathbb{R}^n$ is (path) **connected** if and only if we can join any pair of points in $S$ by a piecewise $\mathcal{C}^1$ path lying completely in $S$. For example, open balls (or boxes) and the interior of an annulus are (path) connected.

**Theorem.** *Let $g$ be a differentiable scalar field with* **continuous** *gradient $\nabla g$ on an open, (path) connected set $\mathcal{D}$ in $R^n$. Then, for any two points $P, Q$ joined by a piecewise $\mathcal{C}^1$ path $C$ completely lying in $\mathcal{D}$ and parametrized by $\alpha : [a, b] \to \mathbb{R}^n$ with $\alpha(a) = P$ and $\alpha(b) = Q$, we have*

$$\int_C \nabla g \cdot d\alpha = g(Q) - g(P).$$

*Remark.* The stunning thing about this result is that the integral depends only on the end points $P$ and $Q$, and not on the path $C$ connecting them. This is pretty revolutionary, if you think about it!

**Corollary.** *Let $g, \mathcal{D}$ be as in the Theorem. Then, for* **any** *point $P$ in $\mathcal{D}$, and* **any** *piecewise $\mathcal{C}^1$ path $C$ connecting $P$ to itself, i.e., with $\alpha(a) = P = \alpha(b)$, we have*

$$\int_C \nabla g \cdot d\alpha = 0.$$

Such a path $C$ whose beginning and end points are the same is called a **closed** path, and the line integral of a vector field $f$ over a closed path $C$ is usually denoted $\oint_C f \cdot d\alpha$.

*Proof of Theorem.* We will prove it for $C$ a $\mathcal{C}^1$ curve, and leave the piecewise $\mathcal{C}^1$ extension, which is routine, to the reader. By definition,

$$\int_C \nabla g \cdot d\alpha = \int_a^b \nabla g(\alpha(t)) \cdot \alpha'(t)\, dt.$$

Put $h(t) = g(\alpha(t))$, for all $t \in [a, b]$. Then by the chain rule, which we can apply since both $\alpha$ and $g$ are differentiable, we have $h'(t) = \nabla g(\alpha(t)) \cdot \alpha'(t)$. So

$$\int_C \nabla g \cdot d\alpha = \int_a^b h'(t)\, dt.$$

Note that $h'(t)$ is continuous on $(a, b)$ by the continuity hypothesis on $\nabla g$. For each $u \in [a, b]$, set $G(u) = \int_a^u h'(t)\, dt$. Then by one variable calculus, $G$ is continuous on $[a, b]$ and differentiable on $(a, b)$ with $G'(u) = h'(u)$ there. So $G - h$ is constant on $(a, b)$ and hence also on $[a, b]$ by continuity. Consequently, since $G(a) = 0$, we have $G(b) = g(\alpha(b)) - g(\alpha(a))$, as needed. □

## 5.3 Criterion for path independence

Let $\mathcal{D}$ be an open, connected set in $\mathbb{R}^n$. The natural question raised by the result of the previous section is to what extent we can characterize vector fields on $f$ on $\mathcal{D}$ whose line integrals are **path independent**, i.e., depend only on the end points. Here is the (extremely satisfying) answer, sometimes called the **first fundamental theorem of Calculus for line integrals**:

**Theorem.** *Let $\mathcal{D}$ be a connected, open set in $\mathbb{R}^n$, and let $f : \mathcal{D} \to \mathbb{R}^n$ be a continuous vector field. Suppose the line integral of $f$ over any piecewise $\mathcal{C}^1$ path $C$ in $\mathcal{D}$ depends only on the end points of $C$. Then there exists a differentiable scalar field $\varphi$ on $\mathcal{D}$, called a* **potential function of** *$f$, such that $\boxed{f = \nabla\varphi}$ on $\mathcal{D}$. Moreover, $\varphi$ can be explicitly defined as follows. Fix any point $P$ in $\mathcal{D}$ and set*

$$\varphi(x) = \int_P^x f \cdot d\alpha,$$

*where the integral is taken over any piecewise $\mathcal{C}^1$ path $C$ in $\mathcal{D}$ connecting $P$ to $x$. Then $\varphi$ is well defined and represents a potential function for $f$.*

Putting the first and second fundamental theorems for line integrals together, we easily obtain the following useful

**Corollary.** *Let $\mathcal{D}$ be an open, connected set in $\mathbb{R}^n$, and let $f : \mathcal{D} \to \mathbb{R}^n$ be a continuous vector field. Then the following properties are* **equivalent***:*

(i) *$f = \nabla\varphi$, for some potential function $\varphi$*

(ii) *The line integrals of $f$ over piecewise $\mathcal{C}^1$ curves in $\mathcal{D}$ are path independent.*

(iii) *The line integrals of $f$ over* **closed***, piecewise $\mathcal{C}^1$ curves in $\mathcal{D}$ are zero.*

**Definition.** A vector field $f$ satisfying any of the (equivalent) properties of Corollary is said to be **conservative**.

*Proof of Theorem.* Let $\varphi$ be defined as in Theorem. That it is well defined is a consequence of the path independence hypothesis for line integrals of $f$ in $\mathcal{D}$. Write $f = (f_1, f_2, \ldots, f_n)$, where $f_j$ is, for each $j \leq n$, the $j$-th component (scalar) field of $f$. We have to show: $(\forall j)$

$$\frac{\partial \varphi}{\partial x_j} \text{ exists and equals } f_j. \tag{$*$}$$

For each $x$ in $\mathcal{D}$, $\exists\, r > 0$ such that the vector $x + he_j$ lies in $\mathcal{D}$ whenever $|h| \in (0, r)$. We may write

$$\varphi(x + he_j) - \varphi(x) = \int_x^{x+he_j} f \cdot d\alpha,$$

where the integral is taken over the line segment $C$ connecting $x$ to $x + he_j$, parametrized by $\alpha(t) = x + the_j$, for $t \in [0, 1]$. Since $\alpha'(t) = he_j$, we get

$$\frac{\varphi(x + he_j) - \varphi(x)}{h} = \int_0^1 f(x + the_j) \cdot e_j \, dt,$$

which equals

$$\begin{aligned} \int_0^1 f_j(x + the_j) \, dt &= \frac{1}{h} \int_0^h f_j(x + ue_j) \, du \\ &= \frac{1}{h} \left( g(h) - g(0) \right), \end{aligned}$$

where $g(t) = \int_0^t f_j(x + ue_j) \, du$, $\forall\, t \in (-r, r)$. Letting $h$ go to zero, we then get the limit $g'(0)$. Thus $\frac{\partial \varphi}{\partial x_j}(x)$ exists and equals $g'(0)$. But by construction, $g'(0) = f_j(x)$, and so we are done.

## 5.4 A necessary condition to be conservative

Not all vector fields $f$ are conservative. It is also quite hard (except in special cases) to check that $f$ is conservative. Luckily, we at least have a test which can be used in many cases to rule out some $f$ from being conservative.

**Theorem.** *Let $f = (f_1, \ldots, f_n)$ be a $\mathcal{C}^1$ vector field on an open, connected set $\mathcal{D}$ in $\mathbb{R}^n$. Suppose $f$ is a conservative field. Then we must have (everywhere on $\mathcal{D}$)*

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \qquad \text{for all } i, j.$$

*Proof.* Suppose $f = \nabla\varphi$, for a potential function $\varphi$ on $\mathcal{D}$, so that $f_j = \frac{\partial\varphi}{\partial x_j}$, for all $j \leq n$. Since $f$ is differentiable, all the partial derivatives of each $f_j$ exist, and we get $(\forall\, i, j)$

$$\frac{\partial f_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial\varphi}{\partial x_j}\right) = \frac{\partial^2\varphi}{\partial x_i \partial x_j}\,,$$

and

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial\varphi}{\partial x_i}\right) = \frac{\partial^2\varphi}{\partial x_j \partial x_i}\,.$$

Since $f$ is $\mathcal{C}^1$, each partial derivative of $f_i$ is continuous for every $i$. Then the mixed partial derivatives $\frac{\partial^2\varphi}{\partial x_i \partial x_j}$ and $\frac{\partial^2\varphi}{\partial x_j \partial x_i}$ are also continuous, and must be equal by an earlier theorem. $\square$

**Examples.** (1) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by $f(x, y, z) = (y, x, 0)$. Denote by $C$ the oriented curve parametrized by $\alpha(t) = (te^t, \cos t, \sin t)$, for $t \in [0, \pi]$. Compute the line integral $I = \int_C f \cdot d\alpha$.

By definition this line integral $I$ is

$$\begin{aligned}\int_0^\pi f(\alpha(t)) \cdot \alpha'(t)\, dt &= \int_0^\pi (\cos t, te^t, 0) \cdot ((1+t)e^t, -\sin t, \cos t)\, dt \\ &= \int_0^\pi [(1+t)\cos t\, e^t - t \sin t\, e^t]\, dt.\end{aligned}$$

This integral can be calculated, but with some trouble. It would have been better had we first checked to see if $f$ could be conservative, i.e., if there is a function $\varphi$ such that $y = \frac{\partial\varphi}{\partial x}$, $x = \frac{\partial\varphi}{\partial y}$ and $0 = \frac{\partial\varphi}{\partial z}$. The last equation says that $\varphi$ is independent of $z$, and the first two can be satisfied by taking $\varphi(x, y, z) = xy$, for all $(x, y, z) \in \mathbb{R}^3$. Thus $f = \nabla\varphi$ everywhere on $\mathbb{R}^3$, and so by the second fundamental theorem for line integrals, $I = \varphi(\alpha(\pi)) - \varphi(\alpha(0)) = \varphi(\pi e^\pi, -1, 0) - \varphi(0, 1, 0) = -\pi e^\pi$.

(2) **(Physics example)** Consider the force field $f(x, y, z) = (x, y, z)$ $(= x\bar{i} + y\bar{j} + z\bar{k})$. Find the work done in moving a particle along the parabola $C = \{(x, y, z) \in \mathbb{R}^3 \mid y = x^2, z = 0\}$ from $x = -1$ to $x = 2$.

We can parametrize $C$ by the $\mathcal{C}^1$ function $\alpha(t) = (t, t^2, 0)$. We are interested in those $t$

lying in the interval $[-1, 2]$. The work done is given by

$$\begin{aligned} W &= \int_C f \cdot d\alpha = \int_{-1}^{2} f(\alpha(t)) \cdot \alpha'(t)\, dt \\ &= \int_{-1}^{2} (t, t^2, 0) \cdot (1, 2t, 0)\, dt = \int_{-1}^{2} (t + 2t^3)\, dt \\ &= \frac{t^2}{2} + \frac{t^4}{2} \Big]_{-1}^{2} = 9. \end{aligned}$$

Alternatively, as in example (1), we could have looked for a potential function $\varphi$ satisfying $\frac{\partial \varphi}{\partial x} = x$, $\frac{\partial \varphi}{\partial y} = y$ and $\frac{\partial \varphi}{\partial z} = z$. An immediate solution is given by $\varphi(x, y, z) = \frac{x^2+y^2+z^2}{2}$. So, by the second fundamental theorem for line integrals,

$$\begin{aligned} W &= \varphi(\alpha(2)) - \varphi(\alpha(-1)) = \varphi(2, 4, 0) - \varphi(-1, 1, 0) \\ &= \frac{2^2 + 4^2}{2} - \frac{(-1)^2 + 1^2}{2} = 9. \end{aligned}$$

(3) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field sending $(x, y, z)$ to $(x^2, xy, 1)$. Determine its line integral over the curve $C$ parametrized by $\alpha(t) = (1, t, e^t)$, for $t \in [0, 1]$.

First let us see if $f$ could be conservative. Since $f$ is differentiable with continuous partial derivatives, we can apply the necessary criterion proved in the previous section. Then, for $f$ to be conservative, we would need $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$, for all $i, j$. Here $f_1(x, y, z) = x^2$, $f_2(x, y, z) = xy$ and $f_3(x, y, z) = 1$. (As usual, we are writing $(x, y, z)$ for $(x_1, x_2, x_3)$.) But $\frac{\partial f_1}{\partial y} = 0$, while $\frac{\partial f_2}{\partial x} = y$. So the criterion fails, and $f$ **cannot** be conservative. We could have also seen this directly by trying to find a potential function $\varphi(x, y, z)$. We would have needed $\frac{\partial \varphi}{\partial x} = x^2$, $\frac{\partial \varphi}{\partial y} = xy$ and $\frac{\partial \varphi}{\partial z} = 1$. The first equation gives (by integrating): $\varphi(x, y, z) = \frac{x^3}{3} + \psi(y, z)$, with $\psi$ **independent of** $x$. Then the second equation forces $\frac{\partial \psi}{\partial y}$, which is $\frac{\partial \varphi}{\partial y}$, to equal $xy$, implying that $\psi$ is **not independent of** $x$. So there is no potential function and $f$ is not conservative.

In any case, we can certainly compute the line integral from first principles:

$$\begin{aligned} \int_C f \cdot d\alpha &= \int_0^1 f(1, t, e^t) \cdot (0, 1, e^t)\, dt \\ &= \int_0^1 (1, t, 1) \cdot (0, 1, e^t)\, dt = \int_0^1 (t + e^t)\, dt \\ &= \left(\frac{t^2}{2} + e^2\right)\Big]_0^1 = \frac{1}{2} + e - 1 = e - \frac{1}{2}. \end{aligned}$$

# Chapter 6

# Green's Theorem in the Plane

## 0 Introduction

Recall the following special case of a general fact proved in the previous chapter. Let $C$ be a piecewise $\mathcal{C}^1$ **plane curve**, i.e., a curve in $\mathbb{R}^2$ defined by a piecewise $\mathcal{C}^1$-function

$$\alpha : [a, b] \to \mathbb{R}^2$$

with **end points** $\alpha(a)$ and $\alpha(b)$. Then for any $\mathcal{C}^1$ scalar field $\varphi$ defined on a connected open set $U$ in $\mathbb{R}^2$ containing $C$, we have

$$\int_C \nabla\varphi \cdot d\alpha = \varphi(\alpha(b)) - \varphi(\alpha(a)).$$

In other words, the integral of the gradient of $\varphi$ along $C$, in the direction stipulated by $\alpha$, depends only on the difference, measured in the right order, of the values of $\varphi$ on the end points. Note that the two-point set $\{a, b\}$ is the boundary of the interval $[a, b]$ in $\mathbb{R}$. A general question which arises is to know if similar things hold for integrals over surfaces and higher dimensional objects, i.e., if the integral of the analog of a gradient, sometimes called an **exact differential**, over a geometric shape $M$ depends only on the integral of its **primitive** on the boundary $\partial M$.

Our object in this chapter is to establish the simplest instance of such a phenomenon for **plane regions**. First we need some preliminaries.

# 1 Jordan curves

Recall that a curve $C$ parametrized by $\alpha : [a, b] \to \mathbb{R}^2$ is said to be **closed** iff $\alpha(a) = \alpha(b)$. It is called a **Jordan curve**, or a **simple closed curve**, iff $\alpha$ is piecewise $\mathcal{C}^1$ and 1-1 (injective) on the open interval $(a, b)$. Geometrically, this means the curve doesn't cross itself. Examples of Jordan curves are **circles**, **ellipses**, and in fact all kinds of **ovals**. The **hyperbola** defined by $\alpha : [0, 1] \to \mathbb{R}^2$, $x \mapsto \frac{c}{x}$, is (for any $c \neq 0$) **simple**, i.e., it does not intersect itself, but it is **not closed**. On the other hand, the **clover** is **closed**, but **not simple**.

Here is a fantastic result, in some sense more elegant than the ones in Calculus we are trying to establish, due to the French mathematician Camille Jordan; whence the name **Jordan curve**.

**Theorem**. *Let $C$ be a Jordan curve in $\mathbb{R}^2$. Then there exists connected open sets $U, V$ in the plane such that*

*(i) $U, V, C$ are pairwise mutually disjoint,*

*and*

*(ii) $\mathbb{R}^2 = U \cup V \cup C$.*

In other words, any Jordan curve $C$ separates the plane into two disjoint, connected regions with $C$ as the common boundary. Such an assertion is obvious for an oval but not (at all) in general. There is unfortunately no way we can prove this magnificent result in this course. But interested students can read a proof in Oswald Veblen's article, "Theory of plane curves in Non-metrical Analysis situs," Transactions of the American Math. Society, **6**, no. 1, 83-98 (1905).

The two regions $U$ and $V$ are called the **interior** or **inside** and **exterior** or **outside** of $C$. To distinguish which is which let's prove

**Lemma:** In the above situation exactly one of $U$ and $V$ is bounded. This is called the interior of $C$.

*Proof.* Since $[a, b]$ is compact and $\alpha$ is continuous the curve $C = \alpha([a, b])$ is compact, hence closed and bounded. Pick a disk $D(r)$ of some large radius $r$ containing $C$. Then $S := \mathbb{R}^2 \setminus D(r) \subseteq U \cup V$. Clearly $S$ is connected, so any two points $P, Q \in S$ can be joined by a continuous path $\beta : [0, 1] \to S$ with $P = \beta(0)$, $Q = \beta(1)$. We have $[0, 1] = \beta^{-1}(U) \cup \beta^{-1}(V)$ since $S$ is covered by $U$ and $V$. Since $\beta$ is continuous the sets $\beta^{-1}(U)$ and $\beta^{-1}(V)$ are open subsets of $[0, 1]$. If $P \in U$, say, put $t_0 = \sup\{t \in [0, 1] | \beta(t) \in U\} \in [0, 1]$. If

$\beta(t_0) \in U$ then, since $\beta^{-1}(U)$ is open, we find points in a small neighborhood of $t_0$ mapped to $U$. If $t_0 < 1$ this would mean we'd find points bigger than $t_0$ mapped into $U$ which contradicts the definition of $t_0$. So if $\beta(t_0) \in U$ then $t_0 = 1$ and $Q = \beta(1) \in U$. If, on the other hand, $\beta(t_0) \in V$, there is an interval of points around $t_0$ mapped to $V$ which also contradicts the definition of $t_0$ (we'd find a smaller upper bound in this case). So the only conclusion is that if one point $P \in S$ lies in $U$ so do all other points $Q$. Then $S \subseteq U$ and $V \subseteq D(r)$ so $V$ is bounded.

Recall that a parametrized curve has an orientation. A Jordan curve can either be oriented counterclockwise or clockwise. We usually orient a Jordan curve $C$ so that the **interior**, $V$ say, lies to the **left** as we traverse the curve, i.e. we take the counterclockwise orientation. This is also called the **positive** orientation. In fact we could define the counterclockwise or positive orientation by saying that the interior lies to the left.

Note finally that the term "interior" is a bit confusing here as $V$ is **not** the set of interior points of $C$; but it is the set of interior points of the union of $V$ and $C$.

# 2 Simply connected regions

Let $R$ be a region in the plane whose interior is connected. Then $R$ is said to be **simply connected** (or s.c.) iff every Jordan curve $C$ in $R$ can be continuously deformed to a point without crossing itself in the process. Equivalently, for **any** Jordan curve $C \subset R$, the interior of $C$ lies completely in the interior of $R$.

**Examples of s.c. regions**:

(1) $R = \mathbb{R}^2$

(2) $R$=interior of a Jordan curve $C$.

The **simplest case of a non-simply connected plane region** is the **annulus** given by $\{v \in \mathbb{R}^2 | c_1 < \|v\| < c_2\}$ with $0 < c_1 < c_2$. The reason is the *hole in the middle* which prevents certain Jordan curves from being collapsed all the way.

When a region is not simply connected, one often calls it **multiply connected** or m.c. The annulus is often said to be *doubly connected* because we

can cut it into two subregions each of which is simply connected. In general, if a region has $m$ holes, it is said to be $(m+1)$-connected, for the simple reason that we can cut it into $m+1$, but no smaller, number of simply connected subregions.

Sometimes we would need a similar, but more stringent notion. A region $R$ is said to be **convex** iff for any pair of points $P, Q$ the line joining $P, Q$ lies entirely in $R$.

A star-shaped region is simply connected but not convex.

# 3 Green's theorem for s.c. plane regions

Recall that if $f$ is a vector field with image in $\mathbb{R}^n$, we can analyze $f$ by its coordinate fields $f_1, f_2, \ldots, f_n$, which are scalar fields. When $n = 2$ (resp. $n = 3$), it is customary notation to use $P, Q$ (resp. $P, Q, R$) instead of $f_1, f_2$ (resp. $f_1, f_2, f_3$).

**Theorem A** (Green) *Let $f = (P, Q)$ be a $\mathcal{C}^1$ vector field on a connected open set $Y$ in $\mathbb{R}^2$. Suppose $C$ is a $\mathcal{C}^1$ Jordan curve with inside (or interior) $U$ such that $\Phi := C \cup U$ lies entirely in $Y$. Then we have*

$$\iint\limits_{\Phi} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint\limits_{C} f \cdot d\alpha$$

Here $C$ is the boundary $\partial\Phi$ of $\Phi$, and moreover, the integral over $C$ is taken in the positive direction. Note that

$$f \cdot d\alpha = (P(x, y), Q(x, y)) \cdot (x'(t), y'(t))dt,$$

and so we are justified in writing

$$\oint\limits_{C} f \cdot d\alpha = \oint\limits_{C} (Pdx + Qdy).$$

Given any $\mathcal{C}^1$ Jordan curve $C$ with $\Phi = C \cup U$ as above, we can try to finely subdivide the region using $\mathcal{C}^1$ arcs such that $\Phi$ is the union of subregions $\Phi_1, \ldots, \Phi_r$ with Jordan curves as boundaries and with non-intersecting insides/interiors, such that each $\Phi_j$ is simultaneously a region of type $I$ and

$II$ (see chapter 5). This can almost always be achieved. So we will content ourselves, mostly due to lack of time and preparation on Jordan curves, with proving the theorem only for regions which are of type $I$ and $II$.

Theorem A follows from the following, seemingly more precise, identities:

(i) $\iint\limits_{\Phi} \frac{\partial P}{\partial x} dxdy = -\oint\limits_{C} Pdx$

(ii) $\iint\limits_{\Phi} \frac{\partial Q}{\partial y} dxdy = \oint\limits_{C} Qdy.$

Now suppose $\Phi$ is of type $I$, i.e., of the form $\{a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\}$, with $a < b$ and $\varphi_1, \varphi_2$ continuous on $[a, b]$. Then the boundary $C$ has 4 components given by

$$\begin{aligned}
C_1 :&\quad \text{graph of } \varphi_1(x),\ a \leq x \leq b \\
C_2 :&\quad x = b, \varphi_1(b) \leq y \leq \varphi_2(b) \\
C_3 :&\quad \text{graph of } \varphi_2(x),\ a \leq x \leq b \\
C_4 :&\quad x = a, \varphi_1(a) \leq y \leq \varphi_2(a).
\end{aligned}$$

As $C$ is positively oriented, each $C_i$ is oriented as follows: In $C_1$, traverse from $x = a$ to $x = b$; in $C_2$, traverse from $y = \varphi_1(b)$ to $y = \varphi_2(b)$; in $C_3$, go from $x = b$ to $x = a$; an in $C_4$, go from $\varphi_2(a)$ to $\varphi_1(a)$.

It is easy to see that

$$\int\limits_{C_2} Pdx = \int\limits_{C_4} Pdx = 0,$$

since $C_2$ and $C_4$ are vertical segments allowing no variation in $x$. Hence we have

(1)

$$-\oint\limits_{C} Pdx = -\left(\int\limits_{C_1} Pdx + \int\limits_{C_3} Pdx\right) = \int\limits_{a}^{b} [P(x, \phi_2(x)) - P(x, \phi_1(x)]dx.$$

On the other hand, since $f$ is a $\mathcal{C}^1$ vector field, $\frac{\partial P}{\partial y}$ is continuous, and since $\Phi$ is a region of type $I$, we may apply Fubini's theorem and get

(2)

$$\iint_{\Phi} \frac{\partial P}{\partial y} dxdy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}.$$

Note that $x$ is fixed in the inside integral on the right. We see, by the fundamental theorem of 1-variable Calculus, that

(3)

$$\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y} = P(x, \varphi_2(x)) - P(x, \varphi_1(x)).$$

Putting together (1), (2) and (3), we immediately obtain identity (i).

Similarly (ii) can be seen to hold by exploiting the fact that $\Phi$ is also of type $II$.

Hence Theorem A is now proved for a region $\Phi$ which is of type $I$ and $II$.

To prove this result for a general region, the basic idea is to cut it into a finite number of subregions each of which is of type I or of type II. For a rigorous treatment of the general situation, read Apostol's *Mathematical Analysis* (chap. 10). For a more geometric point of view, look at Spivak's *Calculus on Manifolds.*

# 4 An area formula

The example below will illustrate a **very useful consequence of Green's theorem**, namely that **the area of the inside of a $\mathcal{C}^1$ Jordan curve** $C$ can be computed as

$$A = \frac{1}{2} \oint_C (xdy - ydx). \tag{*}$$

A proof of this fact is easily supplied by taking the vector field $f = (P, Q)$ in Theorem A to be given by $P(x, y) = -y$ and $Q(x, y) = x$. Clearly, $f$ is $\mathcal{C}^1$ everywhere on $\mathbb{R}^2$, and so the theorem is applicable. The identity for $A$ follows as $\frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y} = 1$.

**Example**:

Fix any positive number $r$, and consider the **hypocycloid** $C$ parametrized by

$$\alpha : [0, 2\pi] \to \mathbb{R}^2, \ t \mapsto (r \cos^3 t, r \sin^3 t).$$

Then $C$ is easily seen to be a piecewise $\mathcal{C}^1$ Jordan curve. Note that it is also given by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = r^{\frac{2}{3}}$. We have

$$A = \frac{1}{2} \oint_C (xdy - ydx) = \frac{1}{2} \int_0^{2\pi} (xy'(t) - yx'(t))dt.$$

Now, $x'(t) = 3r \cos^2 t(-\sin t)$, and $y'(t) = 3r \sin^2 t \cos t$. Hence

$$xy'(t) - yx'(t) = (r \cos^3 t)(3r \sin^2 t \cos t) + (r \sin^3 t)(3r \cos^2 t \sin t)$$

which simplifies to $3r^2 \sin^2 t \cos^2 t$, as $\sin^2 t + \cos^2 t = 1$. So we obtain

$$A = \frac{3r^2}{2} \int_0^{2\pi} \left(\frac{\sin 2t}{2}\right)^2 dt = \frac{3r^2}{8} \int_0^{2\pi} \left(\frac{1 - \cos 4t}{2}\right) dt,$$

by using the trigonometric identities $\sin 2u = 2 \sin u \cos u$ and $\cos 2u = 1 - 2\sin^2 u$. Finally, we then get

$$A = \frac{2r^2}{16} \left[\int_0^{2\pi} (1 - \cos 4t)dt\right] = \frac{3r^2}{16} \left(t - \frac{\sin 4t}{4}\right)\Big]_0^{2\pi}$$

i.e.,

$$A = \frac{3\pi r^2}{8}.$$

# 5 Green's theorem for multiply connected regions

We mentioned earlier that the **annulus in the plane** is the simplest example of a non-simply connected region. But it is not hard to see that we can cut this region into two pieces, each of which is the interior of a $\mathcal{C}^1$ Jordan curve.

We may then apply Theorem A of §3 to each piece and deduce statement over the annulus as a consequence. To be precise, pick any point $z$ in $\mathbb{R}^2$ and consider

$$\Phi = \bar{B}_z(r_2) - \bar{B}_z(r_1),$$

for some real numbers $r_1, r_2$ such that $0 < r_1 < r_2$. Here $\bar{B}_z(r_i)$ denotes the closed disk of radius $r_i$ and center $z$.

Let $C_1$, resp. $C_2$, denote the positively oriented (circular) boundary of $\bar{B}_z(r_1)$, resp. $\bar{B}_z(r_2)$. Let $D_i$ be the **flat diameter** of $\bar{B}_z(r_i)$, i.e., the set $\{x_0 - r_i \leq x \leq x_0 + r_i,\ y = y_0\}$, where $x_0$ (resp. $y_0$) denotes the $x$-coordinate (resp. $y$-coordinate) of $z$. Then $D_2 \cap \Phi$ splits as a disjoint union of two horizontal segments $D_+ = D_2 \cap \{x > x_0\}$ and $D_- = D_2 \cap \{x < x_0\}$. Denote by $C_i^+$ (resp. $C_i^-$) the upper (resp. lower) half of the circle $C_i$, for $i = 1, 2$. Then $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^+$ (resp. $\Phi^-$) is the region bounded by $C^+ = C_2^+ \cup D_- \cup C_1^+ \cup D_+$ (resp. $C^- = C_2^- \cup D_+ \cup D_1^- \cup D_-$). We can orient the piecewise $\mathcal{C}^1$ Jordan curves $C^+$ and $C^-$ in the positive direction. Let $U^+, U^-$ denote the interiors of $C^+, C^-$. Then $U^+ \cap U^- = \emptyset$, and $\Phi^\pm = C^\pm \cup U^\pm$.

Now let $f = (P, Q)$ be a $\mathcal{C}^1$-vector field on a connected open set containing $\bar{B}_z(r_2)$. Then, combining what we get by applying Green's theorem for s.c. regions to $\Phi^+$ and $\Phi^-$, we get:

$$\iint\limits_{\Phi} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint\limits_{C_2} (Pdx + Qdy) - \oint\limits_{C_1} (Pdx + Qdy), \qquad (*)$$

where both the line integrals on the right are taken in the positive (counter-clockwise) direction. Note that the minus sign in front of the second line integral on the right comes from the need to orient $C^\pm$ positively.

In some sense, this is the key case to understand as any multiply connected region can be broken up into a finite union of shapes each of which can be continuously deformed to an annulus. Arguing as above, we get the following (slightly stronger) result:

**Theorem B (Green's theorem for m.c. plane regions)** *Let $C_1, C_2, \ldots, C_r$ be* **non-intersecting** *piecewise $\mathcal{C}^1$ Jordan curves in the plane with interiors $U_1, U_2, \ldots, U_r$ such that*

*(i) $U_1 \supset C_i,\ \forall i \geq 2$,*

*and*

*(ii) $U_i \cap U_j = \emptyset$, for all $i, j \geq 2$.*

*Put* $\Phi = C_1 \cup U_1 - \cup_{i=2}^r U_i$*, which is* **not simply connected** *if* $r \geq 2$*. Also let* $f = (P, Q)$ *be a* $\mathcal{C}^1$ *vector field on a connected open set* $S$ *containing* $\Phi$*. Then we have*

$$\iint\limits_{\Phi} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint\limits_{C_1} (Pdx + Qdy) - \sum_{i=2}^{r} \oint\limits_{C_i} (Pdx + Qdy)$$

*where each* $C_j, j \geq 1$ *is positively oriented.*

**Corollary**: *Let* $C_1, \ldots C_r,\ f$ *be as in Theorem B. In addition, suppose that* $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ *everywhere on* $S$*. Then we have*

$$\oint\limits_{C_1} (Pdx + Qdy) = \sum_{i=2}^{r} \oint\limits_{C_i} (Pdx + Qdy).$$

# 6 The winding number

Let $C$ be an oriented piecewise $\mathcal{C}^1$ curve in the plane, and let $z = (x_0, y_0)$ be a point not lying on $C$. Then the **winding number of $C$ relative to** $z$ is intuitively the number of times $C$ wraps around $z$ in the positive direction. (If we reverse the orientation of $C$, the winding number changes sign.) Mathematically, this number is defined to be

$$W(C, z) := \frac{1}{2\pi} \oint\limits_{C} \left( -\frac{y - y_0}{r^2} dx + \frac{x - x_0}{r^2} dy \right),$$

where $r = \|(x, y) - z\| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.

When $C$ is parametrized by a piecewise $\mathcal{C}^1$ function $\alpha : [a, b] \to \mathbb{R}^2,\ \alpha(t) = (u(t), v(t))$, then it is easy to see that

$$W(C, z) = \frac{1}{2\pi} \int\limits_{a}^{b} \frac{(u(t) - x_0)v'(t) - (v(t) - y_0)u'(t)}{(u(t) - x_0)^2 + (v(t) - y_0)^2} dt$$

Some write $W(\alpha, z)$ instead of $W(C, z)$.

We note the following **useful result** without proof:

**Proposition**: *Let* $C$ *be a piecewise* $\mathcal{C}^1$ *closed curve in* $\mathbb{R}^2$*, and let* $z \in \mathbb{R}^2$ *be a point not meeting* $C$*.*

*(a)* $W(C, z) \in \mathbb{Z}$.

*(b)* $C$ *Jordan curve* $\Rightarrow$ $W(C, z) \in \{0, 1, -1\}$.

*More precisely, in the case of a Jordan curve,* $W(C, z)$ *is 0 if* $z$ *is outside* $C$*, and it equals* $\pm 1$ *if* $z$ *is inside* $C$.

The reader is advised to do the easy verification of this Proposition for the unit circle $C$. In this case, when $z$ is outside $C$, the winding number is zero. When it is inside, $W(C, z)$ is 1 or $-1$ depending on whether or not $C$ is oriented in the positive (counter-clockwise) direction.

Another fun exercise will be to exhibit, for each integer $n$, a piecewise $\mathcal{C}^1$, closed curve $C$ and a point $z$ not lying on it, such that $W(C, z) = n$. Of course this cannot happen if $C$ is a Jordan curve if $n \neq 0, \pm 1$.

*Note*: If $\bar{C}$ denotes the curve obtained from $C$ by reversing the orientation, then the mathematical definition does give $W(\bar{C}, z) = -W(C, z)$.

# CHAPTER 7

## DIV, GRAD, AND CURL

### 1. The operator $\nabla$ and the gradient:

Recall that the gradient of a differentiable scalar field $\varphi$ on an open set $\mathcal{D}$ in $\mathbb{R}^n$ is given by the formula:

$$\nabla\varphi = \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \dots, \frac{\partial\varphi}{\partial x_n}\right). \tag{1}$$

It is often convenient to define formally the differential operator in vector form as:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right). \tag{2}$$

Then we may view the gradient of $\varphi$, as the notation $\nabla\varphi$ suggests, as the result of multiplying the vector $\nabla$ by the scalar field $\varphi$. Note that the order of multiplication matters, i.e., $\frac{\partial\varphi}{\partial x_j}$ is **not** $\varphi\frac{\partial}{\partial x_j}$.

Let us now review a couple of facts about the gradient. For any $j \leq n$, $\frac{\partial\varphi}{\partial x_j}$ is identically zero on $\mathcal{D}$ iff $\varphi(x_1, x_2, \dots, x_n)$ is independent of $x_j$. Consequently,

$$\nabla\varphi = 0 \text{ on } \mathcal{D} \quad \Leftrightarrow \quad \varphi = \text{constant}. \tag{3}$$

Moreover, for any scalar $c$, we have:

$$\nabla\varphi \text{ is normal to the level set } L_c(\varphi). \tag{4}$$

Thus $\nabla\varphi$ gives the direction of steepest change of $\varphi$.

### 2. Divergence

Let $F : \mathcal{D} \to \mathbb{R}^n$, $\mathcal{D} \subset \mathbb{R}^n$, be a differentiable vector field. (Note that *both* spaces are $n$-dimensional.) Let $F_1, F_2, \dots, F_n$ be the component (scalar) fields of $f$. The **divergence of F** is defined to be

$$\text{div}(F) = \nabla \cdot F = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}. \tag{5}$$

This can be reexpressed symbolically in terms of the dot product as

$$\nabla \cdot F = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) \cdot (F_1, \ldots, F_n). \tag{6}$$

Note that $\text{div}(F)$ is a scalar field.

Given any $n \times n$ matrix $A = (a_{ij})$, its **trace** is defined to be:

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Then it is easy to see that, if $DF$ denotes the **Jacobian matrix** of $F$, i.e., the $n \times n$-matrix $(\partial F_i / \partial x_j)$, $1 \leq i, j \leq n$, then

$$\nabla \cdot F = \text{tr}(DF). \tag{7}$$

Let $\varphi$ be a twice differentiable scalar field. Then its **Laplacian** is defined to be

$$\nabla^2 \varphi = \nabla \cdot (\nabla \varphi). \tag{8}$$

It follows from (1),(5),(6) that

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \cdots + \frac{\partial^2 \varphi}{\partial x_n^2}. \tag{9}$$

One says that $\varphi$ is **harmonic** iff $\nabla^2 \varphi = 0$. Note that we can formally consider the dot product

$$\nabla \cdot \nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) \cdot (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \tag{10}$$

Then we have

$$\nabla^2 \varphi = (\nabla \cdot \nabla) \varphi. \tag{11}$$

**Examples of harmonic functions:**

**(i)** $\mathcal{D} = \mathbb{R}^2$; $\varphi(x, y) = e^x \cos y$.

Then $\frac{\partial\varphi}{\partial x} = e^x \cos y$, $\frac{\partial\varphi}{\partial y} = -e^x \sin y$,

and $\frac{\partial^2\varphi}{\partial x^2} = e^x \cos y$, $\frac{\partial^2\varphi}{\partial y^2} = -e^x \cos y$. So, $\nabla^2\varphi = 0$.

**(ii)** $\mathcal{D} = \mathbb{R}^2 - \{0\}$. $\varphi(x, y) = \log(x^2 + y^2) = 2\log(r)$.

Then $\frac{\partial\varphi}{\partial x} = \frac{2x}{x^2+y^2}$, $\frac{\partial\varphi}{\partial y} = \frac{2y}{x^2+y^2}$, $\frac{\partial^2\varphi}{\partial x^2} = \frac{2(x^2+y^2)-2x(2x)}{(x^2+y^2)^2} = \frac{-2(x^2-y^2)}{(x^2+y^2)^2}$, and $\frac{\partial^2\varphi}{\partial y^2} = \frac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$. So, $\nabla^2\varphi = 0$.

**(iii)** $\mathcal{D} = \mathbb{R}^n - \{0\}$. $\varphi(x_1, x_2, ..., x_n) = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\alpha/2} = r^\alpha$ for some fixed $\alpha \in \mathbb{R}$.

Then $\frac{\partial\varphi}{\partial x_i} = \alpha r^{\alpha-1}\frac{x_i}{r} = \alpha r^{\alpha-2}x_i$, and

$\frac{\partial^2\varphi}{\partial x_i^2} = \alpha(\alpha-2)r^{\alpha-4}x_i \cdot x_i + \alpha r^{\alpha-2} \cdot 1$.

Hence $\nabla^2\phi = \sum_{i=1}^n \left(\alpha(\alpha-2)r^{\alpha-4}x_i^2 + \alpha r^{\alpha-2}\right) = \alpha(\alpha-2+n)r^{\alpha-2}$.

So $\phi$ is harmonic for $\alpha = 0$ or $\alpha = 2 - n$ ($\alpha = -1$ for $n = 3$).

## 3. Cross product in $\mathbb{R}^3$

The three-dimensional space is very special in that it admits a **vector product**, often called the **cross product**. Let **i**,**j**,**k** denote the standard basis of $\mathbb{R}^3$. Then, for all pairs of vectors $v = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $v' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$, the cross product is defined by

$$(12)\quad v \times v' = det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ x' & y' & z' \end{pmatrix} = (yz' - y'z)\mathbf{i} - (xz' - x'z)\mathbf{j} + (xy' - x'y)\mathbf{k}.$$

**Lemma 1.** ***(a)*** *$v \times v' = -v' \times v$ (anti-commutativity)*

***(b)*** $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$, $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$, $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$

***(c)*** $v \cdot (v \times v') = v' \cdot (v \times v') = 0$.

**Corollary:** $v \times v = 0$.

**Proof of Lemma** (a) $v' \times v$ is obtained by interchanging the second and third rows of the matrix whose determinant gives $v \times v'$. Thus $v' \times v = -v \times v'$.

(b) $\mathbf{i} \times \mathbf{j} = det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, which is **k** as asserted. The other two identities are similar.

(c) $v \cdot (v \times v') = x(yz' - y'z) - y(xz' - x'z) + z(xy' - x'y) = 0$. Similarly for $v' \cdot (v \times v')$.

Geometrically, $v \times v'$ can, thanks to the Lemma, be interpreted as follows. Consider the plane $P$ in $\mathbb{R}^3$ defined by $v$,$v'$. Then $v \times v'$ will lie along the normal to this plane at the origin, and its orientation is

given as follows. Imagine a corkscrew perpendicular to $P$ with its tip at the origin, such that it turns clockwise when we rotate the line $Ov$ towards $Ov'$ in the plane $P$. Then $v \times v'$ will point in the direction in which the corkscrew moves perpendicular to $P$.

Finally the length $||v \times v'||$ is equal to the area of the parallelogram spanned by $v$ and $v'$. Indeed this area is equal to the volume of the parallelepiped spanned by $v$, $v'$ and a unit vector $u = (u_x, u_y, u_z)$ orthogonal to $v$ and $v'$. We can take $u = v \times v'/||v \times v'||$ and the (signed) volume equals

$$\begin{aligned}\det \begin{pmatrix} u_x & u_y & u_z \\ x & y & z \\ x' & y' & z' \end{pmatrix} =& u_x(yz' - y'z) - u_y(xz' - x'z) + u_z(xy' - x'y) \\ =& ||v \times v'|| \cdot (u_x^2 + u_y^2 + u_z^2) = ||v \times v'||.\end{aligned}$$

## 4. Curl of vector fields in $\mathbb{R}^3$

Let $F : \mathcal{D} \to \mathbb{R}^3$, $\mathcal{D} \subset \mathbb{R}^3$ be a differentiable vector field. Denote by $P$,$Q$,$R$ its coordinate scalar fields, so that $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then the **curl of F** is defined to be:

$$\operatorname{curl}(F) = \nabla \times F = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}. \tag{13}$$

Note that it makes sense to denote it $\nabla \times F$, as it is formally the cross product of $\nabla$ with $f$. Explicitly we have

$$\nabla \times F = (\partial R/\partial y - \partial Q/\partial z)\,\mathbf{i} - (\partial R/\partial x - \partial P/\partial z)\,\mathbf{j} + (\partial Q/\partial x - \partial P/\partial y)\,\mathbf{k}$$

If the vector field $F$ represents the **f**low of a fluid, then the **curl** measures *how the flow rotates the vectors*, whence its name.

**Proposition 1.** *Let $h$ (resp. $F$) be a $\mathcal{C}^2$ scalar (resp. vector) field. Then*

**(a):** $\nabla \times (\nabla h) = 0$.
**(b):** $\nabla \cdot (\nabla \times F) = 0$.

**Proof:** (a) By definition of gradient and curl,

$$\nabla \times (\nabla h) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix}$$

$$= \left(\frac{\partial^2 h}{\partial y \partial z} - \frac{\partial^2 h}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 h}{\partial z \partial x} - \frac{\partial^2 h}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 h}{\partial x \partial y} - \frac{\partial^2 h}{\partial y \partial x}\right)\mathbf{k}.$$

Since $h$ is $\mathcal{C}^2$, its second mixed partial derivatives are independent of the order in which the partial derivatives are computed. Thus, $\nabla \times (\nabla f h) = 0$.

(b) By the definition of divergence and curl,

$$\nabla \cdot (\nabla \times F) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \left(\frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z}\right) + \left(-\frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 P}{\partial y \partial z}\right) + \left(\frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y}\right).$$

Again, since $F$ is $\mathcal{C}^2$, $\frac{\partial^2 R}{\partial x \partial y} = \frac{\partial^2 R}{\partial y \partial x}$, etc., and we get the assertion.

$\square$

**Warning:** There exist **twice differentiable** scalar (resp. vector) fields $h$ (resp. $F$), which are **not** $\mathcal{C}^2$, for which (a) (resp. (b)) does **not** hold.

When the vector field $F$ represents fluid flow, it is often called **irrotational** when its curl is 0. If this flow describes the movement of water in a stream, for example, to be *irrotational* means that a small boat being pulled by the flow will not rotate about its axis. We will see later that the condition $\nabla \times F = 0$ occurs naturally in a purely mathematical setting as well.

**Examples: (i)** Let $\mathcal{D} = \mathbb{R}^3 - \{0\}$ and $F(x, y, z) = \frac{y}{(x^2+y^2)}\mathbf{i} - \frac{x}{(x^2+y^2)}\mathbf{j}$. Show that $F$ is irrotational. Indeed, by the definition of curl,

$$\nabla \times F = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{(x^2+y^2)} & \frac{-x}{(x^2+y^2)} & 0 \end{pmatrix}$$

$$= \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)\mathbf{i} + \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2}\right)\mathbf{j} + \left(\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right)\mathbf{k}$$

$$= \left[\frac{-(x^2+y^2)+2x^2}{(x^2+y^2)^2} - \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\right]\mathbf{k} = 0.$$

**(ii)** Let $m$ be any integer $\neq 3$, $\mathcal{D} = \mathbb{R}^3 - \{0\}$, and $F(x, y, z) = \frac{1}{r^m}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, where $r = \sqrt{x^2 + y^2 + z^2}$. Show that $F$ is not the curl of another vector field. Indeed, suppose $F = \nabla \times G$. Then, since $F$ is $\mathcal{C}^1$, $G$ will be $\mathcal{C}^2$, and by the Proposition proved above, $\nabla \cdot F = \nabla \cdot (\nabla \times G)$ would be zero. But,

$$\nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{x}{r^m}, \frac{y}{r^m}, \frac{z}{r^m}\right)$$

$$= \frac{r^m - 2x^2(\frac{m}{2})r^{m-2}}{r^{2m}} + \frac{r^m - 2y^2(\frac{m}{2})r^{m-2}}{r^{2m}} + \frac{r^m - 2z^2(\frac{m}{2})r^{m-2}}{r^{2m}}$$

$$= \frac{1}{r^{2m}}\left(3r^m - m(x^2+y^2+z^2)r^{m-2}\right) = \frac{1}{r^m}(3-m).$$

This is non-zero as $m \neq 3$. So $F$ is **not** a curl.

**Warning:** It may be true that the divergence of $F$ is zero, but $F$ is still not a curl. In fact this happens in example (ii) above if we allow $m = 3$. We cannot treat this case, however, without establishing Stoke's theorem.

## 5. An interpretation of Green's theorem via the curl

Recall that Green's theorem for a plane region $\Phi$ with boundary a piecewise $\mathcal{C}^1$ Jordan curve $C$ says that, given any $\mathcal{C}^1$ vector field $G = (P, Q)$ on an open set $\mathcal{D}$ containing $\Phi$, we have:

$$\iint_\Phi \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_C P\,dx + Q\,dy. \tag{14}$$

We will now interpret the term $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. To do that, we think of the plane as sitting in $\mathbb{R}^3$ as $\{z = 0\}$, and define a $\mathcal{C}^1$ vector field $F$ on $\tilde{D} := \{(x,y,z) \in \mathbb{R}^3 | (x,y) \in \mathcal{D}\}$ by setting $F(x,y,z) = G(x,y) = P\mathbf{i} + Q\mathbf{j}$. We can interpret this as taking values in $\mathbb{R}^3$ by thinking of its value as $P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k}$.Then $\nabla \times F = det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{pmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$, because $\frac{\partial P}{\partial z} = \frac{\partial Q}{\partial z} = 0$. Thus we get:

$$(\nabla \times F) \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}. \tag{15}$$

And Green's theorem becomes:

**Theorem 1.** $\iint_\Phi (\nabla \times F) \cdot \boldsymbol{k}\, dx\,dy = \oint_C P\,dx + Q\,dy$

## 6. A criterion for being conservative via the curl

A consequence of the reformulation above of Green's theorem using the curl is the following:

**Proposition 1.** *Let $G : \mathcal{D} \to \mathbb{R}^2$, $\mathcal{D} \subset \mathbb{R}^2$ open and* **simply connected**, *$G = (P,Q)$, be a $\mathcal{C}^1$ vector field. Set $F(x,y,z) = G(x,y)$, for all $(x,y,z) \in \mathbb{R}^3$ with $(x,y) \in \mathcal{D}$. Suppose $\nabla \times F = 0$. Then $G$ is conservative on $D$.*

**Proof:** Since $\nabla \times F = 0$, the reformulation in section 5 of Green's theorem implies that $\oint_C P\,dx + Q\,dy = 0$ for all Jordan curves C contained in $\mathcal{D}$. **QED**

**Example:** $\mathcal{D} = \mathbb{R}^2 - \{(x, 0) \in \mathbb{R}^2 \mid x \leq 0\}$, $G(x, y) = \frac{y}{x^2+y^2}\mathbf{i} - \frac{x}{x^2+y^2}\mathbf{j}$. Determine if $G$ is conservative on $\mathcal{D}$:

Again, define $F(x, y, z)$ to be $G(x, y)$ for all $(x, y, z)$ in $\mathbb{R}^3$ such that $(x, y) \in \mathcal{D}$. Since $G$ is evidently $\mathcal{C}^1$, $F$ will be $\mathcal{C}^1$ as well. By the Proposition above, it will suffice to check if $F$ is irrotational, i.e., $\nabla \times F = 0$, on $\mathcal{D} \times \mathbb{R}$. This was already shown in Example (i) of section 4 of this chapter. So $G$ is conservative.

# Chapter 8

# Change of Variables, Parametrizations, Surface Integrals

## §0. The transformation formula

In evaluating any integral, if the integral depends on an auxiliary function of the variables involved, it is often a good idea to change variables and try to simplify the integral. The formula which allows one to pass from the original integral to the new one is called the **transformation formula** (or **change of variables formula**). It should be noted that certain conditions need to be met before one can achieve this, and we begin by reviewing the one variable situation.

Let $\mathcal{D}$ be an open interval, say $(a, b)$, in $\mathbb{R}$ , and let $\varphi : \mathcal{D} \to \mathbb{R}$ be a 1-1 , $\mathcal{C}^1$ mapping (function) such that $\varphi' \neq 0$ on $\mathcal{D}$. Put $\mathcal{D}^* = \varphi(\mathcal{D})$. By the hypothesis on $\varphi$, it's either increasing or decreasing everywhere on $\mathcal{D}$. In the former case $\mathcal{D}^* = (\varphi(a), \varphi(b))$, and in the latter case, $\mathcal{D}^* = (\varphi(b), \varphi(a))$. Now suppose we have to evaluate the integral

$$I = \int_a^b f(\varphi(u))\varphi'(u)\ \mathrm{d}u,$$

for a nice function $f$. Now put $x = \varphi(u)$, so that $\mathrm{d}x = \varphi'(u)\ \mathrm{d}u$. This change of variable allows us to express the integral as

$$I = \int_{\varphi(a)}^{\varphi(b)} f(x)\ \mathrm{d}x = \mathrm{sgn}(\varphi') \int_{\mathcal{D}^*} f(x)\ \mathrm{d}x,$$

where $\mathrm{sgn}(\varphi')$ denotes the sign of $\varphi'$ on $\mathcal{D}$. We then get the transformation formula

$$\int_{\mathcal{D}} f(\varphi(u))|\varphi'(u)|\ \mathrm{d}u = \int_{\mathcal{D}^*} f(x)\ \mathrm{d}x$$

This generalizes to higher dimensions as follows:

**Theorem** *Let $\mathcal{D}$ be a bounded open set in $\mathbb{R}^n$, $\varphi : \mathcal{D} \to \mathbb{R}^n$ a $\mathcal{C}^1$, 1-1 mapping whose Jacobian determinant* $\det(D\varphi)$ *is everywhere non-vanishing on $\mathcal{D}$, $\mathcal{D}^* = \varphi(\mathcal{D})$, and $f$ an integrable function on $\mathcal{D}^*$. Then we have the* **transformation formula**

$$\int \cdots \int_{\mathcal{D}} f(\varphi(u))|\det D\varphi(u)|\ du_1 ... \ du_n = \int \cdots \int_{\mathcal{D}^*} f(x)\ dx_1 ... \ dx_n.$$

Of course, when $n = 1$, $\det D\varphi(u)$ is simply $\varphi'(u)$, and we recover the old formula. This theorem is quite hard to prove, and we will discuss the 2-dimensional case in detail in §1. In any case, this is one of the most useful things one can learn in Calculus, and one should feel free to (properly) use it as much as possible.

It is helpful to note that if $\varphi$ is **linear**, i.e., given by a linear transformation with associated matrix $M$, then $D\varphi(u)$ is just $M$. In other words, the Jacobian determinant is constant in this case. Note also that $\varphi$ is $\mathcal{C}^1$, even $\mathcal{C}^\infty$, in this case, and moreover $\varphi$ is $1-1$ iff $M$ is invertible, i.e., has non-zero determinant. Hence $\varphi$ is $1-1$ iff $\det D\varphi(u) \neq 0$. But this is special to the linear situation. Many of the cases where change of variables is used to perform integration are when $\varphi$ is not linear.

## §1. The formula in the plane

Let $\mathcal{D}$ be an open set in $\mathbb{R}^2$. We will call a mapping $\varphi : \mathcal{D} \to \mathbb{R}^2$ as above **primitive** if it is either of the form

$$(P1) \qquad (u, v) \to (u, g(u, v))$$

or of the form

$$(P2) \qquad (u, v) \to (h(u, v), v),$$

with $\partial g/\partial v, \partial h/\partial u$ nowhere vanishing on $\mathcal{D}$. (If $\partial g/\partial v$ or $\partial h/\partial u$ vanishes at a finite set of points, the argument below can be easily extended.)

We will now prove the transformation formula when $\varphi$ is a composition of two primitive transformations, one of type (P1) and the other of type (P2).

For simplicity, let us assume that the functions $\partial g/\partial v(u, v)$ and $\partial h/\partial u(u, v)$ are always positive. (If either of them is negative, it is elementary to modify the argument.) Put $\mathcal{D}_1 = \{(h(u, v), v) | (u, v) \in \mathcal{D}\}$ and $\mathcal{D}^* = \{(x, g(x, v)) | (x, v) \in \mathcal{D}_1\}$. By hypothesis, $\mathcal{D}^* = \varphi(\mathcal{D})$.

Enclose $\mathcal{D}_1$ in a closed rectangle $R$, and look at the intersection $P$ of $\mathcal{D}_1$ with a partition of $R$, which is bounded by the lines $x = x_m, m = 1, 2, ...,$ and $v = v_r, r = 1, 2, ...,$ with the subrectangles $R_{mr}$ being of sides $\triangle x = l$ and $\triangle v = k$. Let $R^*$, respectively $R^*_{mr}$, denote the image of $R$, respectively $R_{mr}$, under $(u, v) \to (u, g(u, v))$. Then each $R^*_{mr}$ is bounded by the parallel lines $x = x_m$ and $x = x_m + l$ and by the arcs of the two curves $y = g(x, v_r)$ and $y = g(x, v_r + k)$. Then we have

$$\text{area}(R^*_{mr}) = \int_{x_m}^{x_m+l} (g(x, v_r + k) - g(x, v_r))\, \mathrm{d}x.$$

By the mean value theorem of 1-variable integral calculus, we can write

$$\text{area}(R^*_{mr}) = l[g(x'_m, v_r + k) - g(x'_m, v_r)],$$

for some point $x'_m$ in $(x_m, x_m+l)$. By the mean value theorem of 1-variable differential calculus, we get

$$\text{area}(R^*_{mr}) = lk\frac{\partial g}{\partial v}(x'_m, v'_r),$$

for some $x'_m \in (x_m, x_m + l)$ and $v'_r \in (v_r, v_r + k)$. So, for any function $f$ which is integrable on $\mathcal{D}^*$, we obtain

$$\iint\limits_{\mathcal{D}^*} f(x,y)\ \mathrm{d}x\ \mathrm{d}y = \lim_P \sum_{m,r} klf(x'_m, g(x'_m, v'_r))\frac{\partial g}{\partial v}(x'_m, v'_r).$$

The expression on the right tends to the integral $\iint\limits_{\mathcal{D}_1} f(x, g(x,v))\frac{\partial g}{\partial v}(x,v)\ \mathrm{d}x\ \mathrm{d}v$. Thus we get the identity

$$\boxed{\iint\limits_{\mathcal{D}^*} f(x,y)\ \mathrm{d}x\ \mathrm{d}y = \iint\limits_{\mathcal{D}_1} f(x, g(x,v))\tfrac{\partial g}{\partial v}(x,v)\ \mathrm{d}x\ \mathrm{d}v}$$

By applying the same argument, with the roles of $x, y$ (respectively $g, h$) switched, we obtain

$$\boxed{\iint\limits_{\mathcal{D}_1} f(x, g(x,v))\tfrac{\partial g}{\partial v}(x,v)\ \mathrm{d}x\ \mathrm{d}y = \iint\limits_{\mathcal{D}} f(h(u,v), g(h(u,v),v))\tfrac{\partial g}{\partial v}(h(u,v),v)\tfrac{\partial h}{\partial u}(u,v)\ \mathrm{d}u\ \mathrm{d}v}$$

Since $\varphi = g \circ h$ we get by chain rule that

$$\det D\varphi(u,v) = \frac{\partial h}{\partial u}(u,v)\frac{\partial g}{\partial v}(h(u,v),v),$$

which is by hypothesis $> 0$. Thus we get

$$\boxed{\iint\limits_{\mathcal{D}^*} f(x,y)\ \mathrm{d}x\ \mathrm{d}y = \iint\limits_{\mathcal{D}} f(\varphi(u,v))|\det D\varphi(u,v)|\ \mathrm{d}u\ \mathrm{d}v}$$

as asserted in the Theorem.

How to do the general case of $\varphi$? The fact is, we can subdivide $\mathcal{D}$ into a finite union of subregions, on each of which $\varphi$ can be realized as a composition of primitive transformations. We refer the interested reader to chapter 3, volume 2, of "Introduction to Calculus and Analysis" by R. Courant and F. John.

## §2. Examples

(1) Let $\Phi = \{(x,y) \in \mathbb{R}^2 | x^2+y^2 \leq a^2\}$, with $a > 0$. We know that $A = \text{area}(\Phi) = \pi a^2$. But let us now do this by using **polar coordinates.** Put

$$\mathcal{D} = \{(r,\theta) \in \mathbb{R}^2 | 0 < r < a, 0 < \theta < 2\pi\},$$

and define $\varphi : \mathcal{D} \to \Phi$ by

$$\varphi(r,\theta) = (r\cos\theta, r\sin\theta).$$

Then $\mathcal{D}$ is a connected, bounded open set, and $\varphi$ is $\mathcal{C}^1$, with image $\mathcal{D}^*$ which is the complement of a negligible set in $\Phi$; hence any integration over $\mathcal{D}^*$ will be the same as doing it over $\Phi$. Moreover,

$$\frac{\partial\varphi}{\partial r} = (\cos\theta, \sin\theta) \text{ and } \frac{\partial\varphi}{\partial\theta} = (-r\sin\theta, r\cos\theta).$$

Hence

$$\det(D\varphi) = \det\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix} = r,$$

which is positive on $\mathcal{D}$. So $\varphi$ is $1-1$.
By the transformation formula, we get

$$A = \iint\limits_{\Phi} \mathrm{d}x\ \mathrm{d}y = \iint\limits_{\mathcal{D}} |\det D\varphi(r,\theta)|\ \mathrm{d}r\ \mathrm{d}\theta = \int\limits_0^a \int\limits_0^{2\pi} r\ \mathrm{d}r\ \mathrm{d}\theta =$$

$$= \int\limits_0^a r\ \mathrm{d}r \int\limits_0^{2\pi} \mathrm{d}\theta = 2\pi \int\limits_0^a r\ \mathrm{d}r = 2\pi \left.\frac{r^2}{2}\right]_0^a = \pi a^2.$$

We can justify the iterated integration above by noting that the *coordinate function* $(r,\theta) \to r$ on the open rectangular region $\mathcal{D}$ is continuous, thus allowing us to use Fubini.

(2) Compute the integral $I = \int\int_R xdxdy$ where $R$ is the region $\{(r,\phi) \mid 1 \leq r \leq 2,\ 0 \leq \phi \leq \pi/4\}$.

We have $I = \int_1^2 \int_0^{\pi/4} r\cos(\phi) r dr d\phi = \sin(\phi)]_0^{\pi/4} \cdot \left.\frac{r^3}{3}\right]_1^2 = \frac{\sqrt{2}}{2}(8/3 - 1/3) = \frac{7\sqrt{2}}{6}$.

(3) Let $\Phi$ be the region inside the parallelogram bounded by $y = 2x, y = 2x-2, y = x$ and $y = x+1$. Evaluate $I = \iint\limits_{\Phi} xy\ \mathrm{d}x\ \mathrm{d}y$.

The parallelogram is spanned by the vectors $(1,2)$ and $(2,2)$, so it seems reasonable to make the *linear* change of variable

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1 & 2\\ 2 & 2\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

since then we'll have to integrate over the box $[0,1]\times[0,1]$ in the $u-v$-region. The Jacobian matrix of this transformation is of course constant and has determinant $-2$. So we get

$$\begin{aligned}
I &= \int_0^1\int_0^1 (u+2v)(2u+2v)|-2|dudv = 2\int_0^1\int_0^1 (2u^2+6uv+4v^2)dudv \\
&= 2\int_0^1 2u^3/3+3u^2v+4v^2u\Big]_0^1 dv = 2\int_0^1 (2/3+3v+4v^2)dv \\
&= 2(2v/3+3v^2/2+4v^3/3\Big]_0^1) = 2(2/3+3/2+4/3) = 7.
\end{aligned}$$

(4) Find the volume of the cylindrical region in $\mathbb{R}^3$ defined by

$$W = \{(x,y,z)|x^2+y^2 \le a^2, 0 \le z \le h\},$$

where $a, h$ are positive real numbers. We need to compute

$$I = \text{ vol}(W) = \iiint\limits_W \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z.$$

It is convenient here to introduce the **cylindrical coordinates** given by the transformation

$$\varphi : \mathcal{D} \to \mathbb{R}^3$$

given by

$$\varphi(r,\theta,z) = (r\cos\theta, r\sin\theta, z),\ \text{ where } \mathcal{D} = \{0<r<a, 0<\theta<2\pi, 0<z<h\}.$$

It is easy to see $\varphi$ is 1-1, $\mathcal{C}^1$ and onto the interior $\mathcal{D}^*$ of $W$. Again, since the boundary of $W$ is negligible, we may just integrate over $\mathcal{D}^*$. Moreover,

$$|\det D\varphi| = |\det \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}| = r$$

By the transformation formula,

Since the function $(r,\theta,z) \to r$ is continuous on $\mathcal{D}$, we may apply Fubini and obtain

$$I = \int\limits_0^h\int\limits_0^a\int\limits_0^{2\pi} r\ \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}z = \pi a^2 h,$$

which is what we expected.

(5) Let $W$ be the **unit ball** $\overline{B}_0(1)$ in $\mathbb{R}^3$ with center at the origin and radius 1. Evaluate

$$I = \iiint\limits_W e^{(x^2+y^2+z^2)^{3/2}} \; \mathrm{d}x \; \mathrm{d}y \; \mathrm{d}z.$$

Here it is best to use the **spherical coordinates** given by the transformation

$$\psi : \mathcal{D} \to \mathbb{R}^3, \; (\rho, \theta, \phi) \to (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi),$$

where $\mathcal{D} = \{0 < \rho < 1, 0 < \theta < 2\pi, 0 < \phi < \pi\}$. Then $\psi$ is $\mathcal{C}^1$, 1-1 and onto $W$ minus a negligible set (which we can ignore for integration). Moreover

$$\det(D\psi) = \det \begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix} =$$

$$= \cos\phi \det \begin{pmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{pmatrix} - \rho\sin\phi \det \begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{pmatrix} =$$

$$= -\rho^2(\cos\phi)^2\sin\phi - \rho^2(\sin\phi)^3 = -\rho^2\sin\phi.$$

Note that $\sin\phi$ is $> 0$ on $(0, \pi)$. Hence $|\det(D\psi)| = \rho^2 \sin\phi$, and we get (by the transformation formula):

$$I = \int_0^1 \int_0^\pi \int_0^{2\pi} e^{\rho^3}\rho^2 \sin\phi \; \mathrm{d}\rho \; \mathrm{d}\phi \; \mathrm{d}\theta = \int_0^1 e^{\rho^3}\rho^2 \; \mathrm{d}\rho \int_0^\pi \sin\phi \; \mathrm{d}\phi \int_0^{2\pi} \; \mathrm{d}\theta =$$

The function $(\rho, \theta, \phi) \to e^{\rho^3}\rho^2\sin\phi$ is continuous on $\mathcal{D}$, and so we may apply Fubini and perform ietrated integration (on $\mathcal{D}$). We obtain

$$= 2\pi \int_0^1 e^{\rho^3}\rho^2 \; \mathrm{d}\rho \; (-\cos\phi)|_0^\pi = \frac{4\pi}{3}\int_0^1 e^u \; \mathrm{d}u,$$

where $u = \rho^3$

$$= \frac{4\pi}{3} e^u \Big]_0^1 = \frac{4\pi}{3}(e-1).$$

## §3. Parametrizations

Let $n, k$ be positive integers with $k \leq n$. A subset $\Phi$ of $\mathbb{R}^n$ is called a **parametrized k-fold** iff there exist a connected region $T$ in $\mathbb{R}^k$ together with a $\mathcal{C}^1$, 1-1 mapping

$$\varphi : T \to \mathbb{R}^n, \; u \to (x_1(u), x_2(u), ..., x_n(u)),$$

such that $\varphi(T) = \Phi$.

It is called a **parametrized surface** when $k = 2$, and a parametrized curve when $k = 1$.

**Example:** Let $T = \{(u, v) \in \mathbb{R}^2 | 0 \leq u < 2\pi, -\frac{\pi}{2} \leq v < \frac{\pi}{2}\}$. Fix a positive number $a$, and define

$$\varphi : T \to \mathbb{R}^3 \text{ by } \varphi(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where

$$x(u, v) = a \cos u \cos v, \; y(u, v) = a \sin u \cos v, \text{ and } z(u, v) = a \sin v.$$

Then

$$x(u, v)^2 + y(u, v)^2 + z(u, v)^2 = a^2(\cos^2 u \cos^2 v + \sin^2 u \cos^v + \sin^2 v) =$$
$$= a^2(\cos^2 v + \sin^2 v) = a^2.$$

Also, given $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 + z^2 = a^2$ and $(x, y, z) \neq (0, 0, \pm 1)$, we can find $u, v \in T$ such that $x = x(u, v), y = y(u, v)$ and $z = z(u, v)$. So we see that $\varphi$ is $\mathcal{C}^1$, 1-1 mapping onto the **standard sphere** $S_0(a)$ **of radius** $a$ in $\mathbb{R}^3$, minus 2 points. If we want to integrate over the sphere, removing those two points doesn't make a difference because they form a set of content zero.

## §4. Surface integrals in $\mathbb{R}^3$

Let $\Phi$ be a parametrized surface in $\mathbb{R}^3$, given by a $\mathcal{C}^1, 1 - 1$ mapping

$$\varphi : T \to \mathbb{R}^3, \quad T \subset \mathbb{R}^2, \quad \varphi(u, v) = (x(u, v), y(u, v), z(u, v)).$$

(When we say $\varphi \in \mathcal{C}^1$, we mean that it is so on an open set containing T.) Let $\xi = (x, y, z)$ be a point on $\Phi$. Consider the curve $C_1$ on $\Phi$ passing through $\xi$ on which $v$ is constant. Then the tangent vector to $C_1$ at $\xi$ is simply given by $\frac{\partial \varphi}{\partial u}(\xi)$. Similarly, we may consider the curve $C_2$ on $\Phi$ passing through $\xi$ on which $u$ is constant. Then the tangent vector to $C_2$ at $\xi$ is given by $\frac{\partial \varphi}{\partial v}(\xi)$. So the surface $\Phi$ has a **tangent plane** $\mathcal{J}_\Phi(\xi)$ at $\xi$ iff $\frac{\partial \varphi}{\partial u}$ and $\frac{\partial \varphi}{\partial v}$ are linearly independent there. From now on we will assume this to be the case (see also the discussion of tangent spaces in Ch. 4 of the class notes). When this happens at every point, we call $\Phi$ **smooth.** (In fact, for integration purposes, it suffices to know that $\frac{\partial \varphi}{\partial u}$ and $\frac{\partial \varphi}{\partial v}$ are independent except at a set $\{\varphi(u, v)\}$ of content zero.)

By the definition of the cross product, there is a natural choice for a **normal vector to $\Phi$ at $\xi$** given by:

$$\frac{\partial \varphi}{\partial u}(\xi) \times \frac{\partial \varphi}{\partial v}(\xi).$$

**Definition :** Let $f$ be a bounded scalar field on the parametrized surface $\Phi$. The **surface integral of $f$ over $\Phi$,** denoted $\iint\limits_{\Phi} f \, \mathrm{d}S$, is given by the formula

$$\iint\limits_{\Phi} f \, \mathrm{d}S = \iint\limits_{T} f(\varphi(u, v)) || \frac{\partial \varphi}{\partial u}(\xi) \times \frac{\partial \varphi}{\partial v}(\xi) || \, \mathrm{d}u \, \mathrm{d}v.$$

We say that $f$ is **integrable** on $\Phi$ if this integral converges. An important special case is when $f = 1$. In this case, we get

$$\boxed{\mathrm{area}(\Phi) = \iint_T ||\tfrac{\partial\varphi}{\partial u}(\xi) \times \tfrac{\partial\varphi}{\partial v}(\xi)||\ \mathrm{d}u\ \mathrm{d}v}$$

Note that this formula is similar to that for a line integral in that we have to put in a scaling factor which measures how the parametrization changes the (infinitesimal) length of the curve (resp. area of the surface). Note also that unlike the case of curves this formula only covers the case of surfaces in $\mathbb{R}^3$ rather than in a general $\mathbb{R}^n$.

**Example:** Find the area of the standard sphere $S = S_0(a)$ in $\mathbb{R}^3$ given by $x^2 + y^2 + z^3 = a^2$, with $a > 0$. Recall the parametrization of $S$ from above given by

$$\varphi : T \to \mathbb{R}^3\ ,\ \varphi(u, v) = (x(u, v), y(u, v), z(u, v)),$$

$$T = \{(u, v) \in \mathbb{R}^2 | 0 \le u < 2\pi, -\frac{\pi}{2} \le v < \frac{\pi}{2}\},$$

$$x(u, v) = a\cos u \cos v,\ \ y(u, v) = a\sin u \cos v,\ \ z(u, v) = a\sin v.$$

So we have

$$\frac{\partial\varphi}{\partial u} = (-a\sin u\cos v, a\cos u\cos v, 0) \text{ and } \frac{\partial\varphi}{\partial v} = (-a\cos u\sin v, -a\sin u\sin v, a\cos v).$$

$$\Rightarrow \frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u\cos v & a\cos u\cos v & 0 \\ -a\cos u\sin v & -a\sin u\sin v & a\cos v \end{pmatrix} =$$

$$= \det\begin{pmatrix} a\cos u\cos v & 0 \\ -a\sin u\sin v & a\cos v \end{pmatrix}\mathbf{i} - \det\begin{pmatrix} -a\sin u\cos v & 0 \\ -a\cos u\sin v & a\cos v \end{pmatrix}\mathbf{j} +$$

$$+\det\begin{pmatrix} -a\sin u\cos v & a\cos u\cos v \\ -a\cos u\sin v & -a\sin u\sin v \end{pmatrix}\mathbf{k} =$$

$$= a^2\cos u\cos^2 v\mathbf{i} + a\sin u\cos^2 v\mathbf{j} + a^2(\sin^2 u\sin v\cos v + \cos^2 u\sin v\cos v)\mathbf{k} =$$

$$= a^2\cos u\cos^2 v\mathbf{i} + a\sin u\cos^2 v\mathbf{j} + a^2\sin v\cos v\mathbf{k}.$$

$$\Rightarrow ||\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v}|| = a^2(\cos^2 u\cos^4 v + \sin^2 u\cos^4 v + \sin^2 v\cos^v)^{1/2} =$$

$$= a^2(\cos^4 v + \sin^2 v\cos^2 v)^{1/2} = a^2|\cos v|.$$

$$\Rightarrow \text{area}(S) = a^2 \int_0^{2\pi} \mathrm{d}u \int_{-\pi/2}^{\pi/2} |\cos v| \ \mathrm{d}v = 2\pi a^2 \int_{-\pi/2}^{\pi/2} \cos v \ \mathrm{d}v$$

$$\Rightarrow \text{area}(S) = 4\pi a^2.$$

Here is a useful result:

**Proposition:** *Let $\Phi$ be a surface in $\mathbb{R}^3$ parametrized by a $\mathcal{C}^1$, 1-1 function*

$$\varphi : T \to \mathbb{R}^3 \text{ of the form } \varphi(u,v) = (u, v, h(u,v)).$$

*In other words, $\Phi$ is the graph of $z = h(x,y)$. Then for any integrable scalar field $f$ on $\Phi$, we have*

$$\iint_\Phi f \ dS = \iint_T f(u,v,h(u,v)) \sqrt{(\frac{\partial h}{\partial u})^2 + (\frac{\partial h}{\partial v})^2 + 1} \ du \ dv.$$

**Proof.**

$$\frac{\partial \varphi}{\partial u} = (1, 0, \frac{\partial h}{\partial u}) \text{ and } \frac{\partial \varphi}{\partial v} = (0, 1, \frac{\partial h}{\partial v}).$$

$$\Rightarrow \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial h}{\partial u} \\ 0 & 1 & \frac{\partial h}{\partial v} \end{pmatrix} = -\frac{\partial h}{\partial u}\mathbf{i} - \frac{\partial h}{\partial v}\mathbf{j} + \mathbf{k}.$$

$$\Rightarrow ||\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}|| = \sqrt{(\frac{\partial h}{\partial u})^2 + (\frac{\partial h}{\partial v})^2 + 1}.$$

Now the assertion follows by the definition of $\iint_\Phi f \ \mathrm{d}S$.

**Example.** Let $\Phi$ be the surface in $\mathbb{R}^3$ bounded by the triangle with vertices (1,0,0), (0,1,0) and (0,0,1). Evaluate the surface integral $\iint_\Phi x \ \mathrm{d}S$.
Note that $\Phi$ is a triangular piece of the plane $x + y + z = 1$. Hence $\Phi$ is parametrized by

$$\varphi : T \to \mathbb{R}^3 \ , \ \varphi(u,v) = (u, v, h(u,v)),$$

where $h(u,v) = 1 - u - v$, and $T = \{0 \le v \le 1 - u, 0 \le u \le 1\}$.

$$\frac{\partial h}{\partial u} = -1 \ , \ \frac{\partial h}{\partial v} = -1, \text{ and } \sqrt{(\frac{\partial h}{\partial u})^2 + (\frac{\partial h}{\partial v})^2 + 1} = \sqrt{1+1+1} = \sqrt{3}.$$

By the Proposition above, we have:

$$\iint_{\Phi} x \ \mathrm{d}S = \sqrt{3} \int_0^1 \int_0^{1-u} u \ \mathrm{d}u \ \mathrm{d}v = \sqrt{3} \int_0^1 u(1-u) \ \mathrm{d}u = \frac{\sqrt{3}}{6}.$$

There is also a notion of an integral of a vector field over a surface. As in the case of line integrals this is in fact obtained by integrating a suitable projection of the vector field (which is then a scalar field) over the surface. Whereas for curves the natural direction to project on is the tangent direction, for a surface in $\mathbb{R}^3$ one uses the **normal** direction to the surface.

Note that a **unit normal vector** to $\Phi$ at $\xi = \varphi(u, v)$ is given by

$$\mathbf{n} = \frac{\frac{\partial \varphi}{\partial u}(\xi) \times \frac{\partial \varphi}{\partial v}(\xi)}{||\frac{\partial \varphi}{\partial u}(\xi) \times \frac{\partial \varphi}{\partial v}(\xi)||}$$

and that $\mathbf{n} = \mathbf{n}(u, v)$ varies with $(u, v) \in T$. This defines a unit vector field on $\Phi$ called the **unit normal field.**

**Definition:** Let $F$ be a vector field on $\Phi$. Then the **surface integral** of $F$ over $\Phi$, denoted $\iint_{\Phi} F \cdot \mathbf{n} \ \mathrm{d}s$, is defined by

$$\iint_{\Phi} F \cdot \mathbf{n} \ \mathrm{d}S = \iint_{T} F(\varphi(u, v)) \cdot \mathbf{n}(u, v) ||\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}|| \ \mathrm{d}u \ \mathrm{d}v.$$

Again, we say that $F$ is integrable over $\Phi$ if this integral converges.

By the definition of $\mathbf{n}$, we have:

$$\iint_{\Phi} F \cdot \mathbf{n} \ \mathrm{d}S = \iint_{T} F(\varphi(u, v)) \cdot (\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}) \ \mathrm{d}u \ \mathrm{d}v.$$

As in the case of the line integral there is a notation for this integral which doesn't make explicit reference to the parametrization $\varphi$ but only to the coordinates $(x, y, z)$ of the ambient space. If $F = (P, Q, R)$ we write

$$\iint_{\Phi} F \cdot \mathbf{n} \ \mathrm{d}S = \iint_{\Phi} P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy. \tag{1}$$

Here the notation $v \wedge w$ indicates a product of vectors which is bilinear (i.e. $(\lambda_1 v_1 + \lambda_2 v_2) \wedge w = \lambda_1 (v_1 \wedge w) + \lambda_2 (v_1 \wedge w)$ and similarly in the other variable) and antisymmetric (i.e. $v \wedge w = -w \wedge v$, in particular $v \wedge v = 0$). In this sense $\wedge$ is similar to $\times$ on $\mathbb{R}^3$ except that $v \wedge w$ does not lie in the same space where $v$ and $w$ lie. On the positive side $v \wedge w$ can be defined for vectors $v, w$ in any vector space. After these lengthy remarks let's see how this formalism works out in practice. If the surface $\Phi$ is

parametrized by a function $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$ then $dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$ and by the properties of $\wedge$ outlined above we have

$$dy \wedge dz = (\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u})du \wedge dv$$

which is the $x$-coordinate of $\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}$. So the equation (1) does indeed hold if we interpret an integral $\int_T f du \wedge dv$ (over some region $T$ in $\mathbb{R}^2$) as an ordinary double integral $\int_T f dudv$ whereas $\int_T f dv \wedge du$ equals $-\int_T f dudv$. All of this suggests that one should give meaning to $dx, dy, dz$ as elements of some vector space (instead of being pure formal as in the definition of multiple integrals). This can be done and is in fact necessary to a complete development of integration in higher dimensions and on spaces more general than $\mathbb{R}^n$.

# Chapter 9

# The Theorems of Stokes and Gauss

## 1 Stokes' Theorem

This is a natural generalization of Green's theorem in the plane to parametrized surfaces $S$ in 3-space with boundary the image of a Jordan curve. We say that $S$ is **smooth** if every point on it admits a tangent plane.

**Theorem** (Stokes) *Let $S$ be a smooth surface in $\mathbb{R}^3$ parametrized by a $\mathcal{C}^2$, $1-1$ function $\varphi : T \to \mathbb{R}^3$, $(u,v) \mapsto (x(u,v), y(u,v), z(u,v))$, where $T$ is a plane region bounded by a piecewise $\mathcal{C}^1$ Jordan curve $\Gamma$. Denote by $C$ the image of $\Gamma$ under $\varphi$. Let $\mathbf{F} = (P, Q, R)$ be a $\mathcal{C}^1$ vector field on an open set containing $S$ and $C$. Then we have*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \oint_C \mathbf{F} \cdot d\alpha, \tag{*}$$

*where the line integral is taken in the direction inherited via $\varphi$ from the positive direction of $\Gamma$.*

**Proof**. First note that both sides of (*) are linear in $\mathbf{F}$. Since $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, it suffices to show the following:
(i) $\iint\limits_S (\nabla \times P\mathbf{i}) \cdot \mathbf{n} dS = \oint\limits_C P dx$,
(ii) $\iint\limits_S (\nabla \times Q\mathbf{j}) \cdot \mathbf{n} dS = \oint\limits_C Q dy$,
and

(iii) $\iint\limits_S (\nabla \times R\mathbf{k}) \cdot \mathbf{n} dS = \oint\limits_C R dz.$

We will establish (i) and leave the (very similiar) cases (ii) and (iii) as (straight-forward) exercises for the reader.

So let $F = P\mathbf{i}$. Then

$$\nabla \times \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & 0 & 0 \end{pmatrix} = \frac{\partial P}{\partial z}\mathbf{j} - \frac{\partial P}{\partial y}\mathbf{k}. \tag{1}$$

Recall

$$\mathbf{n} dS = \left( \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right) dudv. \tag{2}$$

We now need the following

**Lemma 1**: *Let* $A = (a_1, a_2, a_3),\ B = (b_1, b_2, b_3),$ *and* $C = (c_1, c_2, c_3)$. *Then*

$$A \cdot (B \times C) = \det \begin{pmatrix} a_1 & a_2 & a_c \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

**Proof of Lemma 1**: By definition,

$$\begin{aligned} B \times C &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \\ &= \mathbf{i} \det \begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} - \mathbf{j} \det \begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} + \mathbf{k} \det \begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix}. \end{aligned}$$

So $A \cdot (B \times C) = a_1 \det \begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} - a_2 \det \begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} + a_3 \det \begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix}.$

The assertion follows.

**Proof of Stokes' theorem** (cont$^{\mathbf{d}}$)

Using (1), (2) and Lemma 1, we get

$$(\nabla \times \mathbf{F}) \cdot \left( \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right) = \det \begin{pmatrix} 0 & \frac{\partial P}{\partial z} & \frac{\partial P}{\partial y} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}.$$

Explicitly, we have

(3)

$$(\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}\right) = -\frac{\partial P}{\partial z}\left(\frac{\partial z}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial z}{\partial u}\frac{\partial x}{\partial v}\right) - \frac{\partial P}{\partial y}\left(\frac{\partial y}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)$$
$$= \left(\frac{\partial P}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial u}\right)\frac{\partial x}{\partial u} - \left(\frac{\partial P}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial u}\right)\frac{\partial x}{\partial u}.$$

Put $p(u,v) = P(\varphi(u,v))$. Then by using chain rule, we get

$$\frac{\partial p}{\partial u} = \frac{\partial P}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial u}$$

and

$$\frac{\partial p}{\partial v} = \frac{\partial P}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial v}.$$

Plugging this information into (3) and simplifying, we get

$$(\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}\right) = \frac{\partial p}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial p}{\partial v}\frac{\partial x}{\partial u}. \tag{4}$$

Put

$$P_1(u,v) = p(u,v)\frac{\partial x}{\partial u} \tag{5}$$

and

$$Q_1(u,v) = p(u,v)\frac{\partial x}{\partial v}.$$

It is easy to check, using the product rule, that

$$\frac{\partial}{\partial u}\left(p\frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(p\frac{\partial x}{\partial u}\right) = \frac{\partial p}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial p}{\partial v}\frac{\partial x}{\partial u}. \tag{6}$$

A word of explanation is needed here. The simple formula has resulted from our use of the equality

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{\partial^2 x}{\partial v \partial u},$$

which is justified because $\varphi$ is $\mathcal{C}^2$.

Putting these together, we obtain

$$(\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}\right) = \frac{\partial Q_1}{\partial u} - \frac{\partial P_1}{\partial v} \tag{7}$$

By the definition of the surface integral of $\nabla \times \mathbf{F}$ over $S$ (see Chapter 9), we get (for $\mathbf{F} = P\mathbf{i}$)

$$\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint\limits_T (\frac{\partial Q_1}{\partial u} - \frac{\partial P_1}{\partial v}) du dv. \tag{8}$$

Applying Green's theorem in the plane, we get

$$\iint\limits_T \left(\frac{\partial Q_1}{\partial u} - \frac{\partial P_1}{\partial v}\right) du dv = \oint\limits_\Gamma P_1 du + Q_1 dv. \tag{9}$$

On the other hands, since $C = \varphi(\Gamma)$, we get

(10)

$$\begin{aligned}\oint\limits_C P dx &= \oint\limits_\Gamma P(\varphi(u,v)) \left(\frac{\partial x}{\partial u}\right) du + P(\varphi(u,v)) \left(\frac{\partial x}{\partial v}\right) dv \\ &= \oint\limits_\Gamma P_1 du + Q_1 dv.\end{aligned}$$

This proves (i). □

# 2 Examples

(1) Let $\mathbf{F}(x,y,z) = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$. Show that the integral of $\mathbf{F}$ over any oriented simple closed curve $C$ which is the boundary of a smooth surface $S$ is zero.

Applying Stokes, we get $\oint\limits_C \mathbf{F} \cdot d\alpha = \iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$. Moreover,

$$\begin{aligned}
\nabla \times \mathbf{F} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^z & xe^z & xye^z \end{pmatrix} \\
&= \mathbf{i}\left(\frac{\partial}{\partial y}(xye^z) - \frac{\partial}{\partial z}(xe^z)\right) - \mathbf{j}\left(\frac{\partial}{\partial x}(xye^z) - \frac{\partial}{\partial x}(ye^z)\right) \\
&\quad + \mathbf{k}\left(\frac{\partial}{\partial x}(xe^z) - \frac{\partial}{\partial y}(ye^z)\right) \\
&= \mathbf{i}(xe^z - xe^z) - \mathbf{j}(ye^z - ye^z) + \mathbf{k}(e^z - e^z) = 0.
\end{aligned}$$

Thus $\oint\limits_C F \cdot d\alpha = 0$.

(2) Let $C$ be the intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$. Orient $C$ such that it corresponds to the positive (counterclockwise) direction in the plane $z = 0$. Use Stokes to compute the line integral $I = \oint\limits_C \mathbf{F} \cdot d\alpha$, where $\mathbf{F} = (-y^3, x^3, -z^3)$.

To do this, note first that $C$ is the boundary of the surface $S$ defined by $z = 1 - x - y$, with $(x, y) \in T := \{x^2 + y^2 \leq 1\}$. In other words, $S$ is parametrized by $\varphi(u, v) = (u, v, 1 - u - v)$. Note also that the boundary $\partial T$ of $T$ is the unit circle $\Gamma$ and that $\varphi(\Gamma) = C$. It is easy to see that $S$ is smooth, and that $\varphi$ is 1-1 and $\mathcal{C}^2$. Also, $\mathbf{F}$ is $\mathcal{C}^1$. So we may apply Stokes and get: $I = \iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$, with $\mathbf{n} = \frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v}$. In this case, $S$ is the graph of the function $h(x, y) = 1 - x - y$, and we computed $\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v}$ for such a surface in the proof of the Proposition in §4 of Chapter 9. We found that $\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v} = (-\frac{\partial h}{\partial u}, -\frac{\partial h}{\partial v}, 1)$, which in the present case is (1,1,1). On the other hand,

$$\begin{aligned}
\nabla \times \mathbf{F} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & -z^3 \end{pmatrix} \\
&= \mathbf{i}\left(\frac{\partial}{\partial y}(-z^3) - \frac{\partial}{\partial z}(x^3)\right) - \mathbf{j}\left(\frac{\partial}{\partial x}(-z^3) - \frac{\partial}{\partial z}(-y^3)\right) + \mathbf{k}\left(\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(y^3)\right) \\
&= (3x^2 + 3y^2)\mathbf{k}
\end{aligned}$$

So, $(\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}\right) = 3x^2 + 3y^2$, and we get

$$\begin{aligned} I &= 3 \iint\limits_T (u^2 + v^2) du dv \\ &= 3 \int\limits_0^1 \int\limits_0^{2\pi} r^2 \cdot r dr d\theta \text{ (in polar coordinates)} \\ &= 6\pi \int\limits_0^1 r^3 dr = \frac{3\pi}{2}. \end{aligned}$$

# 3 The Divergence Theorem of Gauss

Let $S$ be a smooth surface in $\mathbb{R}^3$, parametrized by a $\mathcal{C}^1$, 1-1 function $\varphi : T \to \mathbb{R}^3$, $(u, v) \mapsto (x(u, v), y(u, v), z(u, v))$. Recall that $\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}$ gives a normal vector at any $\xi = \varphi(u, v)$. This defines what one calls an **orientation** of $S$, i.e., gives a choice at each point between the two possible normal directions.

A surface $S$ in $\mathbb{R}^3$ is said to be **closed** if it is the boundary of a (3-dimensional) region $V$ in $\mathbb{R}^3$. In this case, we write $\partial V$ for $S$. Note that at any point on a closed surface $\partial V$, are normal points inward, i.e., towards the interior of $V$, and the other points outward.

We say that a smooth surface $S$ is **orientable** if as we move the inward normal along a curve on $S$ and come back to the initial point, then the inward normal continues to remain the inward normal. If we take the rectangular strip and glue the ends to make a **cylinder**, resp. **Möbius band**, then the resulting surface is orientable, resp. non-orientable. Note that a smooth, *closed* surface is orientable, but it has two possible orientations, the first with the normal pointing *inward*, and the second with the normal pointing *outward*.

Now we are in a position to state the divergence theorem.

**Theorem** (Gauss) *Let $V$ be a region in $\mathbb{R}^3$ with boundary $S$, a closed surface, oriented by choosing the unit outward normal* $\mathbf{n}$ *to $S$. Let $\mathbf{F} = (P, Q, R)$ be a $\mathcal{C}^1$ vector field on $V$. Then we have*

$$\iiint\limits_V (\nabla \cdot \mathbf{F}) dx dy dz = \iint\limits_S \mathbf{F} \cdot \mathbf{n} dS \qquad (*)$$

We will now prove this for regions of a very special type. Note first that both sides of (*) are linear in $\mathbf{F}$, so it suffices to establish the following:

(Gi) $\iiint\limits_V \frac{\partial P}{\partial x} dxdydz = \iint\limits_S P\mathbf{i}\cdot\mathbf{n}dS$

(Gii) $\iiint\limits_V \frac{\partial Q}{\partial y} dxdydz = \iint\limits_S Q\mathbf{j}\cdot\mathbf{n}dS$

and

(Giii) $\iiint\limits_V \frac{\partial R}{\partial z} dxdydz = \iint\limits_S R\mathbf{k}\cdot\mathbf{n}dS$

We will **prove (Giii)** under the hypothesis that $V$ is a **region of type I**, i.e., given by $f_1(x,y) \leq z \leq f_2(x,y),\ (x,y) \in T$, where $T$ is a connected region in $\mathbb{R}^2$, and $f_1,\ f_2$ continuous. In this case, using Fubini, we get

$$\iiint\limits_V \frac{\partial R}{\partial z} dxdydz = \iint\limits_T dxdy \left( \int\limits_{f_1(x,y)}^{f_2(x,y)} \frac{\partial R}{\partial z} dz \right)$$
$$= \iint\limits_T [R(x,y,f_2(x,y) - R(x,y,f_1(x,y)]dxdy.$$

$S = \partial V$ is a closed surface whose **bottom** $S_1$, say, is the graph of $z = f_1(x,y),\ (x,y) \in T$, and whose **top** $S_2$, say, is the graph of $z = f_2(x,y),\ (x,y) \in T$. Let us denote by $S_3$ the complement of $S_1 \cup S_2$ in $S$, and by $\mathbf{n}_j$ the unit outward normal on $S_j$, $1 \leq j \leq 3$. Then

$$\iint\limits_S R\mathbf{k}\cdot\mathbf{n}dS = \iint\limits_S R\mathbf{k}\cdot\mathbf{n}_1 dS + \iint\limits_S R\mathbf{k}\cdot\mathbf{n}_2 dS,$$

since $\mathbf{k}\cdot\mathbf{n}_3 = 0$. (This is because $n_3$ lies in the $(x,y)$-plane.) As we saw in the proof of the Proposition in §4 of Chapter 8, we have

$$\mathbf{n}_1 = \frac{\frac{\partial f_1}{\partial x}\mathbf{i} + \frac{\partial f_1}{\partial y}\mathbf{j} - \mathbf{k}}{\sqrt{\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2 + 1}}$$

Note that the sign is fixed by the fact that the **outward** normal $\mathbf{n}_1$ must have negative $\mathbf{k}$-component. In other words,

$$\mathbf{n}_1 dS = \left(\frac{\partial f_1}{\partial x}\mathbf{i} + \frac{\partial f_1}{\partial y}\mathbf{j} - \mathbf{k}\right) dudv \quad \text{and} \quad \mathbf{n}_2 dS = \left(-\frac{\partial f_1}{\partial x}\mathbf{i} - \frac{\partial f_1}{\partial y}\mathbf{j} + \mathbf{k}\right) dudv.$$

Consequently,

$$\mathbf{k}\cdot\mathbf{n}_1 dS = -dudv, \quad \text{and} \quad \mathbf{k}\cdot\mathbf{n}_2 dS = dudv$$

and

$$\iint_{S_1} R\mathbf{k}\cdot\mathbf{n}_1 dS = -\iint_T R(u,v,f_1(u,v))dudv$$

and

$$\iint_{S_2} R\mathbf{k}\cdot\mathbf{n}_1 dS = \iint_T R(u,v,f_2(u,v))dudv.$$

Thus

$$\begin{aligned}\iint_S R\mathbf{k}\cdot\mathbf{n}_1 dS &= \iint_T [R(u,v,f_2(u,v)) - R(u,v,f_1(u,v))]dudv.\\ &= \iint_T \frac{\partial R}{\partial z}dxdydz.\end{aligned}$$

This proves (Giii).

By a similar argument, (Gii) (resp. (Gi)) holds if $V$ is a region of type II (resp. type III), i.e., if it is given by $g_1(x,z) \le y \le g_2(x,z)$, $(x,z) \in T_1$ (resp. $h_1(y,z) \le x \le h_2(y,z)$, $(y,z) \in T_2$), where $T_1$ (resp. $T_2$) is a connected region in $\mathbb{R}^2$ and $g_1, g_2$ (resp. $h_1, h_2$) are continuous.

Thus Gauss's theorem holds at least when $V$ is of type I, II and III simultaneously. (Check, for example, that the ball in $\mathbb{R}^3$ is of this form.) Unfortunately, we have not developed things to a point where we can prove Gauss's theorem without such stupid hypotheses.

# 4 Examples

(1) Let $S = S_0(1)$ be the **unit sphere** $x^2+y^2+z^2 = 1$ in $\mathbb{R}^3$ with center at 0 and radius 1. Compute $I : \iint_S \mathbf{F}\cdot\mathbf{n}dS$, where $\mathbf{F} = (2x, y^2, z^2)$. By Gauss's

theorem, $I = \iiint\limits_V \nabla \cdot \mathbf{F} dxdydz$, where $V$ is the unit ball $\bar{B}_0(1)$. We have:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2 + 2y + 2z.$$

$$\Rightarrow I = 2 \iiint\limits_V (1 + y + z) dxdydz.$$

Note that

$$\iiint\limits_V ydxdydz = \iiint\limits_V zdxdydz = 0$$

by the symmtery of $V$ around 0 and the oddness of the function $y$ (resp. $z$) in the $y$ direction (resp. $z$ direction).

So, $I = 2 \iiint\limits_V dxdydz = 2\text{vol}(V) = \frac{8\pi}{3}$.

The moral of this problem is that one should always try to first exploit the symmetry arguments if present, before grunging out the calculation.

(2) Let $S$ be the surface of the cylinder $x^2 + y^2 = 1$, $-1 < z < 1$, and $x^2 + y^2 \leq 1$ when $z = \pm 1$. Compute $I := \iint\limits_S \mathbf{F} \cdot \mathbf{n} dS$, where $\mathbf{F} = (xy^2, x^2 y, y)$. Again by Gauss's theorem, $I = \iiint\limits_V \nabla \cdot \mathbf{F} dxdydz$, where $V = \{x^2 + y^2 \leq 1,\ -1 \leq z \leq 1\}$. Also,

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(y) = y^2 + x^2.$$

$$\Rightarrow I = \iiint\limits_V (x^2 + y^2) dxdydz = \int_{-1}^{1} dz \left( \iint\limits_{\mathcal{D}_0} (x^2 + y^2) dxdy \right),$$

where $\mathcal{D}_0 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$. Using polar coordinates to evaluate the inside integral we get $\iint\limits_{\mathcal{D}_0} (x^2 + y^2) dxdy = \int_0^{2\pi} d\theta (\int_0^1 r^3 dr) = \frac{\pi}{2}$. Thus $I = \frac{\pi}{2} \int_{-1}^{1} dz = \pi$.